# FOR A QUALITATIVE THEORY OF THE BOUNDARY LAYER (*)

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*ABSTRACT* — A foundation for the boundary layer equations of hydrodynamics is presented based on its mathematical structure. The connection between the physical arguments used for simplifying the Navier-Stokes-Poisson equations and the resulting partial differential equations is shown, allowing a clear definition of boundary layer flows. Comparison theorems are summarized and used in the construction of bounds for the solutions. These form the basis for a qualitative theory of the boundary-layer.

## 0 — INTRODUCTION

Boundary layer theory was founded in 1904 by L. Prandtl [1] in an attempt to reconcile theoretical Hydrodynamics with experimental evidence and to free it from embarassing paradoxes.

That branch of Theoretical Fluid Mechanics, based on Euler equations, is devoid of viscous effects. Viscous effects are taken into account on the Navier-Stokes-Poisson equations.

However, even if Navier-Stokes-Poisson equations are believed to be no more than a first order approximation to a real viscous fluid they present such mathematical difficulties that no complete theory or even existence and uniqueness proofs are known for its solution on the general case.

Boundary layer theory is the result of an approximation to the Navier-Stokes-Poisson equations when the flow has a dominant

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direction and the field variables change at least one order of magnitude faster in the cross stream direction. This is the typical behaviour of an almost paralell flow over a solid boundary of a slightly viscous fluid because: viscous fluid effects are associated with second order derivatives in space, and these are only important near the solid boundary in the direction normal to it; viscous effects are confined to a thin layer, the so called boundary-layer.

## 1. THE BOUNDARY-LAYER EQUATIONS

Formally, the Navier-Stokes-Poisson equations are, in a cartesian frame of reference:

1.1) $$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \rho F_i$$

1.2) $$\frac{\partial \rho}{\rho t} + u_j \frac{\partial \rho}{\partial x_j} = -\frac{\partial u_j}{\partial x_j}$$

where $u_i (i, j = 1, 2, 3)$ are the velocity components; $\rho$ specific mass; P pressure and F a body force per unit mass, (the summation convention on equal indices is used). For the system to become determinate a Thermodynamic relation involving P and $\rho$ is necessary.

If the flow has a dominant direction along $x_1 \equiv x$, and is described in two independent variables $(x \equiv x_1 ; y \equiv x_2)$ and

$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2}$$

The system of equations 1) can be simplified to

1.3) $$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{1}{\rho}\frac{\partial P}{\partial x} + \rho F_x$$

1.4) $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

where $u \equiv u_1 ; v \equiv u_2$

3) and 4) are the boundary layer equations for a viscous compressible fluid. Besides, a Thermodynamic relation between P and $\rho$ is to be added.

The same type of reasoning which originated 3) and 4) in cartesian coordinates can be applied to other sets of coordinates provided the basic assumptions regarding second order space derivatives are retained.

1.1. The simplifications which originated 3) and 4) were introduced by Prandtl on pure physical arguments and a complete discussion on such lines can be found elsewhere [2]. Some attempts to justify mathematically the approximation using the theory of asymptotic solutions of differential equations with respect to a small parameter, have also appeared as well as higher order corrections [3] [4] [5] and specially [6]. As a rule, only the time independent case is considered when discussing the general theory, and only a few exceptions are concerned with the general compressible case.

As it stands today, boundary-layer theory is far from complete in spite of thousands of papers and a dozen or so of textbooks dealing with the subject.

Because the simplications were found by order of magnitude analysis, appealing to physical arguments, inconsistencies arise frequently even in simple situations. Besides, insuficient consideration of the mathematical structure of the equations often originated wrong approaches to solutions or deficient analysis of experimental data, not mentioning confusion in semantics which arise any time a separation occurs or a transition to turbulence appears.

1.2. Although an approximation to the Navier-Stokes-Poisson (NSP) equations, the boundary-layer equations have never been solved exactly in general terms. Generally speaking, it became common use to consider exact solution those solutions which can be found from the integration of an ordinary non-linear differential equation. This forms the family of similar solutions which are easily got numerically.

However, these are not exact in the rigorous sense. They are not general because a similar solution, got through the transformation of a partial differential equations to an ordinary differential one only exists for a particular combination of initial/boundary conditions.

To substantiate previous remarks without to much involvement on a critical review of the whole theory we will stress some particular points. The first concerns the general philosophy of the approximation; the other two, common inconsistencies systematically overlooked.

The original argument of Prandtl was that viscous effects would be confined to a thin layer near the solid boundary over which the flow passes. Prandtl neglected curvature effects and considered the normals to the boundary as normals to the streamlines. This is the usual approach. However, an optimal coordinate system can be found where displacements of the flow can be taken into account and the approximation to the full set of the N. S. P. equations improved [7]. Following this approach, each flow situation would give rise to a unique form of the boundary layer equations. This would be the first step in the construction of a consistent theory. Such a treatment has not been followed and different equations (all aiming at the description of the same physical phenomena and all starting from the full N. S. P. equations) are obtained depending of the coordinate system used.

As a second remark on the order of magnitude arguments in common use we can consider the flow over a flat plate when the free stream velocity $U_G$ is not constant.

Because $U \to U_G$ when $y \to \infty$ it follows from the continuity equation that

$$v = \int_0^y - \frac{\partial u}{\partial x} \, dy$$

because $\frac{\partial u}{\partial x} \neq 0$, and $\frac{\partial u}{\partial x} = \frac{\partial u_G}{\partial x}$ at the edge of the boundary layer, it follows that $v(x, \infty)$ becomes unbounded, which obviously can not be true. Besides, the assumption that $v << u$ is a common argument in deriving the boundary layer equations.

The arguments based in $v << u$ are unsound as is again verified in the case of a circular jet discharging into stagant surroundings. Even the classical similar solution shows that $v >> u$ in the outer region of the jet.

The quoted examples are but a few. They do not invalidate the practical value of the theory though both point out for a need of clarification regarding the foundations of the theory and its limitations. If too much overlooked the approximate analytical results are useless and with them the predictive value of the theory.

1. 4. Recognizing the enormous practical value of boundary layer theory, and the difficulty in obtaining analytical solutions, the purpose of this contribution is two-fold:

— To clarify the basic structure of the mathematical theory.
— To provide analytic bounds for the exact solutions.
— To formulate the basis for the devellopment of general numerical methods of solution.

## 2 — SOME RESULTS FROM THE THEORY OF NON-LINEAR PARABOLIC EQUATIONS

2. 1. Consider the non-linear equation

2. 1) $$\varphi\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0 .$$

2. 2) If $$\frac{\partial \varphi}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} < 0$$

for a solution $u(x, y)$, equation 1. 1) is parabolic.

If

2. 3) $$\mathrm{L}[u] \equiv a\left(x, y, u, \frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial x} = f\left(x, y, u, \frac{\partial u}{\partial y}\right)$$

with $a > 0$.

Equation 1. 3) is a quasi-linear parabolic equation.

THEOREM 1 — [8].

Let,

2. 4) $$\mathrm{L}[u] = f\left(x, y, u, \frac{\partial u}{\partial y}\right) \qquad \text{in } \mathrm{D} + \mathrm{B_T}$$

under the initial condition

2.5) $$u(x,0) = \psi(y) \quad \text{in } \overline{B}$$

and the boundary condition

2.6) $$u(x,y) = g(x,y) \quad \text{in S}$$

and assume that $f(x,y,u,w)$, where $w = \frac{\partial u}{\partial y}$ is defined for

$$(x,y) \in \overline{D} \qquad -\infty < u < \infty, \qquad -\infty < \frac{\partial u}{\partial y} < \infty$$

Then, if L is parabolic with continuous bounded coefficients in $D + B_T$, and $f(x,y,u,w)$ is monotone nondecreasing in $u$, there exists at most one solution to 2.4), 2.5), 2.6).

*Obs*: By a solution $u$ we mean a continuous function in $\overline{D} (\equiv D + B_T + \overline{B} + S)$ having two continuous $y$ derivatives and one continuous $x$ derivative in $D + B_T$ such that 2.4), 2.5), 2.6), are satisfied.

THEOREM 2 (Comparaison Theorem) [9].

Let

2.7) $$L[u] = f\left(x, y, u, \frac{\partial u}{\partial y}\right) \quad \text{in } D + B_T.$$

Suppose that $u$ is a solution of

$$L[u] = f \quad \text{in } D + B_T$$

$$u(y,0) = g_1(y) \quad \text{in } \overline{B}$$

2.8) $$u(x,y) = g_2(x,y) \quad \text{in S}$$

and assume that $z$ and Z satisfy the inequalities

2.9) $$L[Z] < f(x,y) < L[z]$$

$$z(y,0) < g_1(y) < Z(y,0) \quad \text{in } \overline{B}$$

2.10) $$z < g_2 < Z \quad \text{in S.}$$

If L is parabolic with respect to the functions

$$\theta u + (1-\theta) \cdot z \text{ and } \theta u + (1-\theta) \cdot Z \qquad \text{for } 0 < \theta < 1,$$

Then

2.11) $$z(x,y) < u(x,y) < Z(x,y).$$

THEOREM 3: [8], [9].

Theorem 2 remains valid for boundary conditions of the type

2.12) $$\frac{\partial u}{\partial n} + \beta(x,y,u) = \varphi$$

if

2.13) $$\frac{\partial Z}{\partial n} + \beta < \frac{\partial u}{\partial n} + \beta < \frac{\partial Z}{\partial n} + \beta$$

and $\frac{\partial}{\partial n}$ exists.

## 3 — THE MATHEMATICAL STRUCTURE OF THE BOUNDARY LAYER EQUATIONS

3.1. If we consider the general time dependent N. S. P. equations they are «parabolic in time»; in steady state they are elliptic [10]. The assumptions formulated by Prandtl for the physical situation considered, changed the equations to «parabolic in $x$». This means, physically, that downstream do not affect what happens upstream.

The implications of dropping second order derivatives in the flow direction on the argument that they are negligible compared with the ones in the cross stream direction is a physical argument which in turn makes dowstream points unable to influence their upstream ones. This is reflected also on the boundary conditions:

retaining second order derivatives in $x$ would originate an elliptic set of equations whose solution would imply specification of velocity values at the most downstream section.

This interconnection of the physical assumptions with the mathematical type of the resulting equations is too often overlooked with consequences on the interpretation of results or the choice of the most suitable solution method.

Because a mathematical classification gives a natural and unique way for dealing with the resultant equations we will consider boundary-layers only those physical situations where the parabolic character of the equations is preserved.

Adopting this definition, which is consistent with the always refered Prandtl example, the simplification of the N. S. P. equations will always be clear whatever the coordinate system used. This will be imposed by the geometry and conditions of the particular problem.

3. 2. Having in mind the remarks above we will restrict the analysis to wall flows over a flat plate. The main conclusions will remain valid for other situations.

The equations are

$$3.1)\qquad \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\,\frac{\partial \mathrm{P}}{\partial x}+\frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right)$$

$$3.2)\qquad \frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0\,.$$

From equation 2. 2) we can consider

$$v=f(u)$$

and 3. 2) is parabolic on the sense of the definition if $u>0$, because $\mu>0$ on account of the second principle of Thermodynamics.

Equations 3. 1) 3. 2) must be solved under the conditions

$$3.3)\qquad u(y\,,o)=\mathrm{U}_1\qquad\text{initial condition}$$

$$3.4)\qquad u\,(x\,,o)=0$$

3.5) $v(x, o) = 0$ boundary conditions

3.6) $u(x, y) \to U_G$

$y = \infty .$

The boundary condition 3.5) can be changed to $v(x, o) = f(x)$, which corresponds to suction or blowing. We can also deal very easily with this case, as will become apparent.

3.3. The set 3.1-3.6, on which general agreement exists, deserves some further comments.

As a consequence of the order of magnitude analysis due to Prandtl the second momentum equation on the steady N. S. P. equations was reduced (in absence of body forces) to

3.7) $$\frac{\partial P}{\partial y} = 0$$

the unknowns were, however, $u, v, P$. As a result of the simplification one equation is lost. The needed information is recovered if $\frac{\partial P}{\partial x}$ is known. This knowledge is got from the solution of the inviscid fluid equations (Euler equations). This is consistent with Prandtl views of confining viscous effects to a thin layer and matching the flow field in this layer with the «inviscid» one outside it. This matching, however, is not obvious in what regards the surface at which it must be achieved. The specification in 3.6 is the usual in boundary layer theory, and is correct as long as $u \to U_G$ though arbitrary in puting it at $y = \infty$. This is at the origin of the difficulty refered to in § 1.4.

Formally it can only be removed increasing the order of approximation in the framework of assymptotic solutions. In principle, it seems that the difficulty can be avoided solving simultaneously the inviscid equations or proceeding by successive approximations in the sense that an inviscid flow field is calculated neglecting viscosity; this flow field allows $U_G$ to be specified for boundary layer equations and a new field, near the wall, is found which will be introduced as a perturbation on the initial inviscid field and so on. It must be stressed that such a procedure will always introduce a discontinuity in $\frac{\partial u}{\partial y}$ which, however, can be made negligeble in physical terms.

Practically, that matching can only be achieved by numerical methods in general situations. In that case the matching is tentatively assumed at a finite distance, $\delta$, and the final position found by iteration.

3.4. If $\frac{\partial P}{\partial x} > 0$ it will be shown that a finite distance always exists where $\frac{\partial u}{\partial y} = 0$ at the wall. The same situation can arise if $\frac{\partial P}{\partial x} > 0$ only on a finite interval provided it assumes strong enough values.

The condition $\frac{\partial u}{\partial y} = 0$, if the conditions on $\frac{\partial P}{\partial x}$ continue to prevail, separates a region of the flow where $u < 0$. The surface $u = 0$ defines a surface of separation if it encloses a domain of $u < 0$. Obviously, in this domain the boundary layer equations are not valid. Physically, the order of magnitude argument which justified the simplifications is no longer valid; mathematically the equations have no more meaningful solutions (the situation is similar to a heat equation with a negative diffusivity). So, both physical and mathematical arguments preclude the use of the boundary layer approximations in separated regions. Nevertheless it became frequent in aeronautics to force its use when dealing with aerofoils [12], and this practice is in the origin of some pitfalls. The situations where such use can be justified are those where the separated region is so small (separation buble) that the flow inside it can be approximated empirically. Because such a buble does not disturb too much the flow outside it, the boundary layer equations are used on the outside and its solution matched with the empirical description of the flow in the buble. It must be stressed, however, that such techniques can only be justified in particular circunstances suported by experiment.

## 4 — VON MISES TRANSFORMATION

4. 0. Introducing a stream function $\psi$ such that

4. 1) $$\psi = \int_0^y \rho u \, dy$$

gives

4. 2) $$u = \frac{\partial \psi}{\partial y}$$

$$v = - \frac{\partial \psi}{\partial x}$$

and 3. 2. is automatically verified.

Taking now $(x, \psi)$ as new coordinates we get the following relations for changing derivatives.

4. 3) $$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial x}\right) - \rho v \frac{\partial}{\partial \psi}$$

4. 4) $$\left(\frac{\partial}{\partial y}\right)_x = \rho u \frac{\partial}{\partial \psi}$$

and the boundary layer equations transform into

4. 5) $$\frac{\partial u}{\partial x} = \frac{\rho_G U_G}{\rho u} \cdot \frac{d U_G}{d x} + \frac{\partial}{\partial \psi}\left(\mu \rho u \frac{\partial u}{\partial \psi}\right)$$

and any other equations like energy or species would tranform on a similar way.

4. 1. The transformation $(x, y)$ to $(x, \psi)$ is only one to one if

$$\frac{\partial (x, \psi)}{\partial (x, y)} \neq 0$$

which is always true if $u > 0$; but $u > 0$ for $y > 0$ is always imposed by the boundary layer approximation, and the condition is verified. This is an advantage regarding other types of approximation like the many times used of Croco which needs $\frac{\partial u}{\partial y} > 0$.

The main drawback to V. Mises coordinates which is usually stressed and seems to have obscured its real advantages is due to a singularity at $y = 0$.

At $y = 0, u = 0$ and 4.5) gives

$$0 = \infty$$

$$\text{also, at } y = 0, \frac{\partial (x, \psi)}{\partial (x, y)} = 0.$$

The dificulty of the singularity at $y = 0$ can be removed in two different ways.

One is assuming that the boundary condition

$$u = 0 \quad \text{for} \quad y = 0$$

is changed to:

$$u = \varepsilon_n \quad \text{for} \quad y = 0$$

where $\varepsilon_n$ is a small, positive quantity.

For the laminar case, Oleinik has proved that the solutions $U_n = U_n(\varepsilon_n)$ tend to a limit when $\varepsilon_n \to 0$. The existence of the limit allows one, practically, to solve for sucessively small $\varepsilon$ and extrapolate to zero.

4.2. The singularity can be removed or avoided. To avoid it, we use the V. M. transformation only for $y > \varepsilon$, with $\varepsilon$ arbitrarily small, and change accordingly the boundary conditions.

Using the b. l. equations in form 3.1, 3.2, and assuming that (it can be proved) near the wall $u$ can be develloped in a power series

4.6) $$U = Ky + by^2 + cy^3 \dots$$

a standard procedure will show that

4. 7) $$b = \frac{\rho_G}{\rho} U_G \frac{d U_G}{d x}$$

$$c = 0$$

. . . . . . . . . . .

Assuming for the moment that $K \neq 0$ and that $y$ is small enough we have

$$U \simeq K y$$

and

4. 8) $$\psi = \int_0^y \rho u \, d v$$

4. 9) $$\psi = \frac{\rho k y^2}{2}$$

4. 10) $$u = \left( \frac{2 \psi}{\rho K} \right)^{1/2}$$

4. 11) $$\frac{\partial u}{\partial \psi} = (2 \psi \rho K)^{-1}$$

and eliminating K from 4. 10)

4. 12) $$\frac{\partial u}{\partial \psi} = \frac{u}{2 \psi}.$$

The boundary condition on Von Mises coordinates is fixed at $\psi_1$ small enough for 4. 9) to be valid (which is always possible because $y$ can be choosed as small as we wish) we will have for the V. M. equations the boundary condition.

4. 13) $$\frac{\partial u}{\partial \psi} = \frac{\partial u}{2 \psi_1} \quad \text{for} \quad \psi = \psi_1$$

the domain in $\psi$ beeing now $\psi > \psi_1$.

The above reasoning shows that the singularity was avoided changing a boundary condition from a Dirichelet to a Fourier type.

As will be noted, $\psi_1$ is a constant which was chosen «a priori».

After solving the equation, $U = U(x, \psi)$ is got and $U_1 = U(x, \psi_1)$.

From $U_1$ and $\psi_1$, K is found from

$$K = \frac{2\,\psi_1}{\rho\, u_1^2} \tag{4.14}$$

and

$$y_1 = \frac{u_1}{K} \tag{4.15}$$

from which the physical coordinates are recovered through

$$y = \frac{u_1}{K} + \int_{\psi_1}^{\psi} \frac{d\,\psi}{\rho\, u}. \tag{4.16}$$

For a not too small $y$ or when $K \simeq 0$ the reasoning can be extended taking now two or more terms in 4. 6.

Formally, the limit when $\psi_1 \to 0$, can also be considered.

Physically, the boundary is moving because $\psi_1$ and not $y_1$ is fixed. As a consequence, when $K \to 0$, $y_1 \to \infty$ which obliges to take two terms in 4. 6) when $K = 0$ (at separation).

If, at separation, 4. 6) is still accepted, the V. M. equations can be applied to the upper part of the boundary layer where backflow is not present.

Using now theorem 2, and the form just presented for the boundary conditions we can show that the solution is unique. That the solution exists can be proved from the numerical solution method to be presented on a following paper. All the conclusions apply, however, if backflow is not present.

In the previous results it must be stressed that:

— It was assumed that no backflow is present $u > 0$); This condition is necessary for the normal parabolicity of the equations.

— No special form was assumed for the viscosity, neither for $\rho$, except that they must be positive, continuous and bounded, which is assured by its physical meaning.

## 5 — THE COORDINATES $(x, \tau)$

51. A new set of coordinates which emerged from a numerical integration procedure will be introduced now, because it posseéses some interesting analytical properties.

The purpose of the transformation is to reduce the unbounded coordinate $y$, or $\psi$ to a finite value.

Define

5.1) $$\tau = 1 - y^{-C\psi}$$

where $C > 0$ is some function of $x$.

$\tau$ and $\psi$ possess a one to one relation and the property

5.2) $$\tau = 0 \quad \text{for} \quad \psi = 0$$
$$\tau = 0 \quad \text{for} \quad \psi = \infty$$
$$\psi = -\frac{1}{C}\log(1-\tau).$$

For small $C\psi$:

5.3) $$\tau \simeq C\psi + \cdots$$

which means that for small enough $C, \tau$ and $\psi$ are proportional. This range depends, of course, on the value of $C\psi$, and is as large as we please in practical terms because $C$ is free.

As one can easily conclude, the new coordinate $\tau$ has the property of bringing $\infty$ to 1 and, through the choice of $C$, the possibility of streching or reducing the coordinates near the wall where the more important phenomena take place.

In the new coordinates, the diferential operators in V. M. coordinates takes the form:

5.4) $$\left(\frac{\partial}{\partial x}\right)_\psi = \left(\frac{\partial}{\partial x}\right)_\tau - \frac{1-\tau}{C}\log(1-\tau)\frac{dC}{dx}\left(\frac{\partial}{\partial \tau}\right)_x$$
$$\left(\frac{\partial}{\partial \psi}\right)_x = C(1-\tau)\frac{\partial}{\partial \tau}$$

and the b. l. equations become, in the new set of coordinates

5.5)
$$\frac{\partial u}{\partial x} - \frac{1-\tau}{C} \log(1-\tau) \frac{dc}{dx} \frac{\partial u}{\partial \tau} = \frac{\rho_G U_G}{\rho u} \frac{du_G}{dx} + $$
$$+ C^2 (1-\tau) \frac{\partial}{\partial \tau} \mu \rho u (1-\tau) \frac{\partial u}{\partial \tau}$$

with the initial and boundary condition:

5.6)
$$U(\tau, 0) = U_1$$

5.7)
$$\frac{\partial u}{\partial \tau} = \frac{C(1-\tau_1)}{u} \qquad \text{for } \tau = \tau_1$$
$$U \to U_G \qquad \text{when } \tau \to 1$$

the b. c., 5.6) is just the expression of the initial profile in the new coordinates.

Boundary condition 5.7) results from a reasoning similar to that used for the V. M. equations to avoid the singularity at the wall.

The last boundary condition was written only as a remainder because it was needed in the previous coordinates. Indeed, in the new coordinates, is automatically verified because

$$\frac{\partial v}{\partial \tau} \neq \infty, \quad \lim_{\tau \to 0} (1-\tau) \log(1-\tau) = 0$$

and $\frac{1}{C} \frac{dC}{dx} \neq \infty$

5.2. A more convenient form of 5.5) can be introduced writing it in terms of the kinetic energy for unit mass:

$$V = \frac{1}{2} u^2 .$$

Introducing this variable the b. l. equations become:

5.8)
$$\frac{\partial V}{\partial x} - \frac{1-\tau}{C} \log(1-\tau) \frac{dc}{dx} \frac{dV}{d\tau} = \frac{\rho_G}{\rho} \frac{dV_G}{dx} + $$
$$+ C^2 \sqrt{2V} (1-\tau) \frac{\partial}{\partial \tau} \mu \rho (1-\tau) \frac{\partial V}{\partial \tau}$$

and the boundary condition transforms into

$$5.9)\qquad \frac{\partial V}{\partial \tau} = -\frac{V}{(1-\tau_1)\log(1-\tau_1)} \quad \text{for} \quad \tau = \tau_1 .$$

As in the previous case, $\tau_1$ is an arbitrary but small quantity, so small that in the physical coordinates

$$U \simeq K y .$$

This means that 5.9) can be written as

$$5.10)\qquad \frac{\partial V}{\partial \tau} = \frac{V}{\tau_1}$$

or bringing K explicitly to the equation

$$5.11)\qquad \frac{\partial V}{\partial \tau} = \frac{K}{\rho C} .$$

This relation is exact on the limit as $\tau_1 \to 0$.

Because this classifies also the way in which the singularity on the Von Mises coordinates has been avoided a brief proof is given:

The identity

$$\frac{\partial u}{\partial y} = \rho u \frac{\partial u}{\partial \psi} = \rho u c (1-\tau) \frac{\partial u}{\partial \tau}$$

can be written

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial \psi}\left(\frac{1}{2} u^2\right) = \rho C (1-\tau) \frac{\partial}{\partial \tau}\left(\frac{1}{2} u^2\right)$$

or

$$\frac{\partial u}{\partial y} = C(1-\tau)\frac{\partial V}{\partial \tau}$$

in the limit when $y \to 0$ $\tau \to 0$ and

$$\frac{\partial u}{\partial y} = \mathrm{K} = \rho\, \mathrm{C} \frac{\partial \mathrm{V}}{\partial \tau}$$

or

$$\frac{\partial \mathrm{V}}{\partial \tau} = \frac{\mathrm{K}}{\rho \mathrm{C}}$$

which shows that the transformation does not introduce any singularity at the wall in what regards the first derivative.

5. 3. Noticing that C is still an unspecified function of $x$, and that $\mathrm{K} = \left(\frac{\partial \mathrm{V}}{\partial y}\right)_{y=0}$ is also a function of $x$, we can take

$$\mathrm{C}(x) = \frac{\mathrm{K}(x - \varepsilon)}{\rho \mathrm{A}}$$

where A is a constant, and $\varepsilon$ some positive function which is zero at $x = 0$. With this choice, $\mathrm{K}(x - \varepsilon)$ is known at each $x$, because at $x = 0$ $\mathrm{K}(0)$ is part of the data. And because the equation is parabolic K is known marching downstream.

In the limit of $\varepsilon = 0$ we have

$$\frac{\partial \mathrm{V}}{\partial \tau} = \mathrm{A}$$

which shows that the problem can be transformed into one of constant derivative as boundary condition.

This form has some peculiar advantages in what concerns analytical studies, and numerical methods as will be shown. The disadvantage is that K and C become unknown unless one of them is found. For this, we must consider an assimptotic expression near $\tau = 0$ and find, for example, the higher order derivatives.

## 6 — ANALYTICAL BOUNDS FOR THE SOLUTIONS

6.1. Many results can be extracted from Theorem 2 and some other from the general theory of partial differential equations.

So far, only the cases of $\frac{dU_G}{dx} > 0$ (favorable pressure gradients) have been studied in this way. The theorems proved by Oleinik, Velte, Serrin, apply only for the case of laminar wall flows without a velocity maximum ($\frac{\partial u}{\partial y} > 0$ for $y > 0$ at the initial station) which excludes wall jets and turbulent flows.

Because some recent results obtained by Serrin [11] on the assumptions refered and $u > 0$ on the whole flow can be obtained from Theorem 2 and 3 (allowing $u = 0$ at $y = 0$) and because they will be useful later on, we quote them without proof.

6.2. Assume that

6.1) $$\frac{dU_G}{dx} > 0$$

6.2) $\tilde{u}(y)$ is the initial profile

6.3) $u(x,y)$ is the solution of the b. l. eq.

6.4) $\frac{\partial u}{\partial y}$ is continuous for $0 < x < \infty$, $0 < y < \infty$.

If:

6.5) *a*) $$U_G(x) = C(x+d)^m \qquad 0 < x < \infty$$
$$C > 0, \ d > 0$$

and $\bar{u}(x,y)$ is the similar solution corresponding to the streaming speed:

$$\bar{u}(x,y) = U(x) f(\eta).$$

Then

$$6.6)\quad \left| \frac{u}{U_G} - f' \right| = 0\left(\frac{1 + m \log x}{u^m}\right) \text{ as } x \to \infty \text{ uniformly in } y.$$

*b*) If $u$ and $\bar{u}$ are two solutions corresponding to the same $U_G(x)$ but different profiles, $\bar{u}$ and $\bar{u}$, the streaming speed is twice continuously differentiable and obeys:

$$6.7)\qquad C_T (x+d)^{2m-1}\, U_G \frac{d U_G}{d x}\, C_2 (x+d)^{2n-1}$$

where $C_1$ and $C_2$ are positive constants and $m$ and $n$ are exponents satisfying the condition

$$6.8)\qquad m < n < \frac{5}{3} m.$$

Then

$$6.9)\qquad \left| \frac{\bar{u} - u}{U_G} \right| = 0(1) \text{ as } x \to \infty \text{ uniformly in } y.$$

6.3. For the skin friction the following results apply: If $\tilde{u}(y)$ is an arbitrary initial profile and $\bar{u}(x, y)$ is the similar solution corresponding to the same streaming speed $(U_G(x) = C(x+d)^m$, then

$$6.10)\qquad \frac{g}{U_G} |\zeta - \bar{\zeta}| = 0\left(\frac{\mu}{x^{2m}}\right) \quad \text{as} \quad x \to \infty$$

where

$$g = \sqrt{\frac{\nu (x+d)}{U_G}}.$$

That is:

The normalized skin friction $\frac{g\zeta}{U_G}$ is assimptotically unique as $x$ tends to infinity.

6.4. The results presented before are precise because some specific assumptions have been made concerning the pressure gradient and because similar solutions exist. The results also imply $\mu = \text{const.}$ and laminar flows.

Now we only assume that $\mu$ is not a function of $x$ (for example, $\mu$ is given by a mixing length hipotesis) and $\rho = \text{const.}$

6. 4. 1. Let us consider a very crude but ilustrative estimate. Put $U_G \frac{d U_G}{d u} = - \frac{d P}{d x}$ to bring the pressure gradient explicitly into account.

Assume as comparison function $Z = f(x)$.

Because Z only depends on $x$, all derivatives in U disappear and the momentum equation gives

$$\frac{df}{dx} = - \frac{1}{f} \frac{dP}{dx}$$

or

$$\frac{d}{dx} f^2 = - \frac{2}{\rho} \frac{dP}{dx}$$

so

$$f^2(x) = f^2(0) - \int_0^x \frac{2}{\rho} \frac{dP}{dx} dx$$

or

$$f = f(0) - 2 \int_0^x \frac{1}{\rho} \frac{dP}{dx} dx$$

and $f$ is an upper bound to the exact solution if

$$f(0) > \max U(0, \psi)$$

because $u$ must always be positive, $f^2 > 0$ is an imposition for the solution to exist.

So, we have the condition

$$U_0^2 - 2 \int_0^x \frac{1}{\rho} \frac{dP}{dx} dx > 0$$

for a solution to exist.

Consider now an adverse pressure gradient: $\frac{dP}{dx} > 0$.

Then, an $x_p$ exists such that:

$$\int_0^{x_p} \frac{1}{\rho} \frac{dP}{dx} = U_0^2$$

so, at this $x_p$ occurs the separation of the boundary layer if it had not yet occured.

This very crude application of the theorem cleary shows that an adverse pressure gradient applied for a sufficient distance always gives separation; and an upper downstream limit for it to occur is given above.

This result is true regardless of the boundary layer being laminar or turbulent and shows the unsoundeness of empirical correlations and semi-empirical results quoted in textbooks for a limit adverse pressure gradient which gives rise to separation. This result also shows the nonphysical character of the so called separating profile in the Falkner-Skan family of similar solutions: only for the corresponding initial profile and absence of arbitrarily small perturbations can such a solution exist.

6.2.2. In equation 4.5) put $V = \frac{1}{2} U^2$, and derive in $\psi$. Putting $Z = \frac{\partial V}{\partial \psi}$ we get

$$\frac{\partial Z}{dx} = Z \frac{\partial}{\partial \psi} (R Z) + V \frac{\partial^2}{\partial \psi^2} (R Z)$$

which is again a parabolic equation.

Putting $Z = f_1(x)$, we get as in 6.2.1.

$$f_1 = \text{const.}$$

and we conclude, using Theorem 3, that in Von Mises coordinates $\frac{\partial V}{\partial \psi} = u \frac{\partial u}{\partial \psi}$ has its maximum and minimum values on the initial line or at the wall

$$\left(\text{because } \frac{\partial u}{\partial \psi} = 0 \text{ when } \psi \to \infty\right).$$

6. 2. 3. Deriving now 4. 5 in order to $x$ we have the conclusion that $\frac{du}{dx}$ has its maximum value on the initial line or on $\psi = \infty$, the maximum rate of decrease (or increase) being $\frac{du_G}{dx}$.

6. 2. 4. For 5. 8), 5. 9) with

$$\frac{\partial V}{\partial \tau} = A$$

we reach the conclusion regarding the $(x, \tau)$ coordinates that

$$\frac{\partial V}{\partial \tau} < A$$

unless on the initial line $\frac{\partial V}{\partial \tau} > A$ for some point. In this case that value is the maximum attainable.

Because $A > 0$, if $\frac{\partial V}{\partial \tau} = B < 0$ on the initial line

$$B < \frac{\partial V}{\partial \tau} < A$$

on the whole flow.

## CONCLUSION

A consistent theory of the boundary layer can be established in connection with its classification as a non-linear parabolic equation on the proper sense. This excludes regions of backflow which is again consistent with the physical concept that downstream events do not influence upstream ones. It is also consistent with the primitive Prandtl analysis. Using this basic framework existence and uniqueness of solution can be proved based on comparaison theorems. These allows, also, the construction of analytical bounds for the exact solution which form the basis of a qualitative theory of the boundary layer. Similar solutions form a class of natural bounds for laminar flows.

For turbulent flows, such bounds depend on the specific forms assumed for the Reynolds stresses.

A general result applicable to both laminar and turbulent flows implies that an adverse pressure gradient always gives rise to separation at a finite distance.

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