MODERN TRENDS IN RESEARCH ON WAVES IN FLUIDS, PART II: PROPAGATION AND DISSIPATION IN COMPRESSIBLE AND IONIZED ATMOSPHERES

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ABSTRACT — We consider the propagation and dissipation of waves in fluids, in the presence of external force fields (§ 1), namely the magnetic and gravity fields: the former in connection with Alfvèn-gravity waves (§ 2) in an ionized atmosphere, including the damping by Ohmic electric resistance; the latter in connection with acoustic waves of large amplitude (§ 3), which tend to shock formation, delayed by diffusive processes such as viscosity. The propagation and dissipation of waves in atmospheres is an effective physical process of transfering mass, momentum and energy; analogous problems of engineering interest occur in the propagation of sound in ducts of varying cross-section, such as the horns of loudspeakers and the nozzles of jet engines.

1 — INTRODUCTION

Two of the main trends in current research on waves in fluids are the study of the effects of internal inhomogeneities and turbulence, which were discussed in the first part of the present essay, and the consequences of external applied force fields, which we consider in the present, concluding part. Gravity, being intrinsically associated with the existence of matter, is an ever present force field, which causes fluids to become stratified, and affects considerably waves in the oceans (Stoker 1953, Eckart

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1960, Philips 1960, Kraus 1977) and atmosphere (Yih 1965, Beer 1974, Hines 1974, Gossard & Hooke 1975, Pedlosky 1979) of the earth. More than a few miles away from the earth's surface magnetic fields are important, both in the mantle and ionosphere, and ionized matter is predominant in most regions of the universe (Chapman & Cowling 1949, Spitzer 1952, Akasofu & Chapman 1972, Parker 1979), the basic properties of waves in these conditions being a standard topic in magnetohydrodynamics (Alfvèn 1948, Landau & Lifshitz 1956, Cowling 1957, Ferraro & Plumpton 1961, Alfvèn & Falthammar 1962, Cabannes 1970, Priest 1982).

The Alfvèn speed, which characterizes the propagation of transverse (and hence incompressible) hydromagnetic waves, scales on the inverse square root of the mass density, and thus increases rapidly with altitude in an atmosphere. The acceleration of the Alfvèn-gravity waves with altitude implies that the wave forms are *not* sinusoidal, neither for standing nor for propagating modes; also, the phase of propagating waves, instead of increasing linearly with altitude, has an increasing *slope*, and tends to a finite asymptotic value. The effects of dissipation, say, by electrical resistance, on the Alfvèn-gravity wave, are significant at low altitude, where it propagates slowly, and become less important at higher altitudes, as the wave speed increases, and there is less time to dissipate its energy; thus there is a transition layer between the low-altitude, diffusion regime and the high-altitude, propagation regime.

Acoustic-gravity waves have rather different properties, since the sound speed is determined by temperature alone, and thus varies slowly. As the atmospheric density decays with altitude, the amplitude of the acoustic-gravity wave increases, until non-linear effects become important and shocks may form. The steepening of the wave front that leads to shock formation is opposed and delayed by dissipation effects, such as viscosity. Even if the static viscosity is small, the kinematic viscosity, which varies inversely with the mass density, may be significant at high-altitude, and thus be a moderately effective dissipation mechanism. Thus acoustic-gravity waves can cause significant compression and mass transport in an atmosphere, together with moderate heating; the Alfvèn-gravity waves, which can be dissipated by electrical resistance, are a more effective mechanism of energy transport and heating, but carry no net mass flux since they are transversal. Another contrast is that the combination of compressibility and gravity can cause the appearance of cut-off frequencies for acoustic waves, whereas Alfvèn waves are not affected by filtering effects.

2 — DAMPING OF HYDROMAGNETIC WAVES WITH VARIABLE SPEED

Waves of small amplitude in homogeneous media are described by linear partial differential equations with constant coefficients, e.g., for magnetic (Alfvèn 1942, Lehnert 1952, Lighthill 1960) and magneto-acoustic (Herlofson 1950, Jones 1964, Campos 1977) waves, and spectral methods can be used in space and time (Lighthill 1978, Adam 1982, Campos 1984a). In the case of stratified media, e.g., atmospheres, the equations describing small amplitude waves are linear with variable coefficients, and exact solutions can be found usually in terms of special functions, e.g., for Alfvèn-gravity (Ferraro 1955, Hide 1956, Ferraro & Plumpton 1958, Zhugzhda 1971, Hollweg 1972, Leroy 1982, Campos 1983b, d) and magnetosonic-gravity (Nye & Thomas 1976, Adam 1977, Campos 1983b, c) waves. We choose among these, as an example, Alfvèn waves propagating in an ionized atmosphere under a vertical external magnetic field B, in the presence of electrical resistance $1/\sigma \neq \infty$ where $\sigma \neq 0$ is the Ohmic conductivity. The only propagating components of the velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$ perturbations are horizontal and parallel $\mathbf{v} = v\,\mathbf{e}_x$, $\mathbf{h} = h\,\mathbf{e}_x$, they depend on altitude z and time t, and satisfy the equation of momentum and induction, viz.:

$$\partial v/\partial t - B^{-1} \{ A(z) \}^2 \, \partial h/\partial z = 0, \tag{1a}$$

$$\partial h/\partial t - B \; \partial v/\partial z = \zeta(z) \; \partial^2 h/\partial z^2, \tag{1b}$$

where the Alfvèn speed $A(z)$ and magnetic diffusivity $\zeta(z)$ generally depend on altitude:

$$\{ A(z) \}^2 = \mu B^2 / 4\pi\rho(z), \quad \zeta(z) = c_*^2 / 4\pi\mu\sigma(z), \tag{2a, b}$$

through the density stratification $\rho(z)$ and Ohmic conductivity $\sigma(z)$; we have denoted by μ the magnetic permeability and by c_* the speed of light.

Eliminating between (1a, b) respectively for the velocity v and magnetic field h perturbations, we obtain the wave equations:

$$\{ \partial^2/\partial t^2 - \partial/\partial z \; (A(z))^2 \; \partial/\partial z - \zeta(z) \; \partial^3/\partial t \, \partial z^2 \} \; h(z,t) = 0, \tag{3a}$$

$$\{ \partial^2/\partial t^2 - (A(z))^2 \; \partial^2/\partial z^2 - (A(z))^2 \; \partial/\partial z \; \zeta(z) \; \partial/\partial z \; (A(z))^{-2} \; \partial/\partial t \} \; v(z,t) = 0; \tag{3b}$$

it is clear that: (i) in an homogeneous medium, for which (2a, b) are constants, the two wave variables satisfy the same equation:

$$\{ \partial^2/\partial t^2 - A^2 \; \partial^2/\partial z^2 - \zeta \; \partial^3/\partial t \, \partial z^2 \} \; v, h(z,t) = 0, \tag{4}$$

and have the same altitude dependence; (ii) in a stratified medium the magnetic field (3a) and velocity (3b) satisfy different equations, as suggested by (1a), and their waveforms are generally different. Resistive Alfvèn-gravity waves are characterized by four parameters, namely the frequency ω which is conserved if the atmosphere is at rest, the scale height $L(z) \equiv \{ d(\log \rho)/dz \}^{-1}$ specifying the density stratification $\rho(z)$, the Alfvèn speed $A(z)$ and the magnetic diffusivity $\zeta(z)$, with which we can form two dimensionless parameters:

$$\alpha(z) \equiv \omega L(z)/A(z), \qquad \beta(z) \equiv \omega \zeta(z)/\{ A(z) \}^2, \tag{5a, b}$$

which may be interpreted as follows: (i) $\alpha \sim 2\pi \lambda/L$ is the compactness since it compares the wavelength λ to the atmospheric scale height L, and describes non-dissipative waves; (ii) in the presence of electrical resistance an additional dissipation parameter $\beta \sim 2\pi \chi/L$ appears, comparing the damping scale $\chi \equiv \zeta/A$ to the scale height.

The Ohmic electrical conductivity σ, and hence the magnetic diffusivity ζ (2b) are independent of density and are functions of temperature, so that they are bounded in a non-isothermal atmosphere (provided the temperature be finite); the mass density, on the other hand, tends to zero $\rho \to 0$ as altitude tends to infinity $z \to \infty$, and thus the Alfvèn speed (2a) diverges asymptotically $A \to \infty$. From the second terms of (3a, b), since $A^2 \, \partial^2 v/\partial z^2$, $A^2 \, \partial h/\partial z$ must remain finite as $z \to \infty$ and $A(z) \to \infty$, and we have $\partial^2 v/\partial z^2$, $\partial h/\partial z \to o$, i.e.: (i) Alfvèn-gravity waves, with electrical resistance or not, propagating in a non-isothermal atmosphere (of bounded temperature), have a velocity perturbation which grows linearly and a magnetic field perturbation which tends to a constant in the asymptotic regime as $z \to \infty$:

$$v(z;\omega) \sim f_1(\omega)\, z + f_2(\omega)\,, \tag{6a}$$

$$h(z;\omega) \sim i\,(B/\omega)\; f_1(\omega)\,, \tag{6b}$$

where f_1, f_2 depend on frequency ω; (ii) in the case of a standing mode perfectly reflected from infinity the velocity perturbation is bounded and thus the magnetic field perturbation decays to zero:

$$v(z;\omega) \sim f_2(\omega)\,, \quad h(z;\omega) \sim 0\,. \tag{7a, b}$$

Thus, although initially the velocity and magnetic field perturbations are proportional $v/A \sim h/B$, in agreement (1a) with equipartition of kinetic and magnetic energy $\rho v^2/2 \sim \rho\,(A^2/B^2)\, h^2/2 \sim \mu h^2/8\pi$, asymptoticaly: (i) for standing modes (7a, b) both the kinetic $E_v = \rho v^2/2 \sim o$ and magnetic $E_h = \mu h^2/8\pi \sim o$ energies (per unit volume) tend to zero; (ii) for propagating waves (6a, b) only the kinetic energy $E_v = \rho v^2/2 \sim O(z^2\, e^{-z/L}) \to o$ since the density decays exponentially on altitude $\rho \sim O(e^{-z/L})$, but the magnetic energy $E_h = \mu h^2/8\pi \sim \mu B^2 f_1^2/8\pi\omega^2$ tends to a constant, so that asymptoticaly all energy is magnetic, i.e., the opposite of equipartition.

The preceding results can be checked in the case of an isothermal atmosphere, for which the wave fields can be calculated exactly at all altitudes and frequencies. In an isothermal atmosphere the density decays exponentially $\rho(z) = \rho(o)\, e^{-z/L}$ on the scale height $L \equiv RT/g$, and the Alfvèn speed (2a) and compactness (5a) are given by:

$$A(z) = \{\mu/4\pi\rho(o)\}^{1/2}\; B\, e^{z/2L} \equiv a\; e^{z/2L}\,, \tag{8a}$$

$$\alpha(z) = (\omega L/a)\; e^{-z/2L} \tag{8b}$$

Considering non-dissipative Alfvèn-gravity waves ($\zeta = o$ in 2b), and using as variable the compactness ($2\,\alpha$ given by 8b) instead of altitude z, the velocity perturbation spectrum $V(z;\omega)$ at altitude z for a wave of frequency ω is shown to satisfy a Bessel equation of order zero. The solution is a linear combination of Hankel functions $H_o^{(1)}(2\alpha)$, $H_o^{(2)}(2\alpha)$ but if we want to select a wave propagating upward, i.e., in the direction of increasing z and decreasing α (8b), we must select $H_o^{(2)}$. The constant of integration is determined from the initial velocity spectrum $V(0;\omega)$ at altitude $z = 0$, and we have:

$$V(z;\omega) = V(o;\omega) \left\{ H_o^{(2)}\left((2\omega L/a)\ e^{-z/2L}\right) / H_o^{(2)}(2\omega L/a) \right\}; \quad (9)$$

for the exact wave field at all altitudes. The solution (9) specifies the transition between the initial and asymptotic regimes, respectively of exponential and linear growth:

$$V(z;\omega) = V(o;\omega)\ e^{z/4L}\ e^{i(\omega/a)z} \left\{ 1 + O\left(\omega z^2/2aL\right) \right\}, \quad (10a)$$

$$V(z;\omega) = \left\{ V(o;\omega)/H_o^{(2)}(2\omega L/a) \right\} \left\{ 1 + \psi - i\, 2z/\pi L \right\} \left\{ 1 + O\left((\omega^2 L^2/a^2)\ e^{-z/L}\right) \right\}, \quad (10b)$$

where $\psi \equiv (2/\pi) \log(\omega L/a) + \phi$, and ϕ is Euler's constant.

In the case of Alfvèn-gravity waves perfectly reflected from infinity, the velocity perturbation spectrum $V(z;\omega)$ must be bounded as $z \to \infty$, and the solution is given by (9) with the Hankel functions of second kind $H_o^{(2)}$ replaced by Bessel functions of first kind J_o. The vanishing of the denominator $J_o(2\omega L/a) = o$ corresponds to resonance, so that the roots p_n of the Bessel function $J_o(p_n) = o$ specify through $p_n = 2\omega L/a = 4\pi L/\lambda$ the frequencies ω_n and wavelengths λ_n of the standing modes:

$$\omega_n = a\, p_n/2L, \quad \lambda_n = 4\pi L/p_n, \quad (11a, b)$$

$$\omega_n/\omega_1 = \lambda_1/\lambda_n = p_1/p_n; \quad (11c)$$

it will be noted that the frequencies of the eigenmodes depend on the Alfvèn speed and scale height (11a), the wavelengths depend only on the scale height (11b), and their ratios are absolute non-integral numbers (11c) independent of wave or atmospheric properties. The velocity perturbation is the superposition of all standing modes with non-vanishing initial spectrum $V(0;\omega_n)$:

$$v(z,t) = (\pi a/2L) \sum_{n=1}^{\infty} \operatorname{Im}\{V(0;\omega_n)\exp(-i\omega_n t)\} \; q_n^{-1} J_o(p_n e^{-z/2L}), \tag{12}$$

where $q_n \equiv J'_o(p_n) = -J_1(p_n)$ is the slope of the Bessel function J_o at its zero p_n. The exact solution (12) is valid at all altitudes, and shows the transition between the initial and asymptotic regimes:

$$v(z,t) = (a/L)(\pi/2)^{1/2} e^{z/4L} \sum_{n=1}^{\infty} \operatorname{Im}\{V(o;\omega_n) e^{-i\omega_n t}\} \; q_n^{-1} p_n^{-1/2} \sin\{(\omega/a)(z-2L)\}\{1 + O(\omega z^2/2aL)\}, \tag{13a}$$

$$v(z,t) = (\pi a/2L) \sum_{n=1}^{\infty} \operatorname{Im}\{V(o;\omega_n) e^{-i\omega_n t}\} q_n^{-1} \{1 + O((p_n^2/4) e^{-z/L})\}, \tag{13b}$$

respectively of exponential growth and finite amplitude.

In the presence of resistive dissipation, bearing in mind that the magnetic diffusivity is bounded (2b) and the Alfvèn speed unbouded as $z \to \infty$ (2a), the dissipative wave equations (3a, b) have at most three regular singularities: (i) the singularity at $z = o$ specifies the initial wave field; (ii) the singularity at $z = \infty$ specifies the asymptotic radiation field; (iii) an intermediate singularity at $z = z_*$ specifies a transition layer where the effects of propagation and diffusion balance:

$$\omega \zeta(z_*) = \{A(z_*)\}^2; \quad z_* = L \log(\omega \zeta/a^2), \tag{14a, b}$$

specifies the altitude of the transition layer for an isothermal atmosphere, where the magnetic diffusivity ζ is constant and the Alfvèn-speed $A(z)$ increases exponentially with altitude according to (8a) from the initial value $a \equiv A(o)$. The transition layer separates the atmosphere in two regions: (i) in the low-altitude region $z < z_*$ the magnetic diffusivity predominates, the wave equations resemble the Schrodinger's type, and we have mainly resistive dissipation modified by propagation; (ii) in the high-altitude region $z > z_*$ the Alfvèn speed predominates, the wave equations resemble the hyperbolic type, so that we have propagation with damping which decays with altitude, and preserves the form of the asymptotic laws (6, 7a, b). The situation is similar to the plane, isentropic, compressible flow, for which the transition is specified by local Mach number unity $M = 1$, separating subsonic flow for which the hodograph equations are elliptic from supersonic flow for which they are hyperbolic (Molenbroek 1890, Chaplygin 1904, Lighthill 1947, von Mises 1960); another case is wave absorption at critical levels (Bretherton 1966, Booker & Bretherton 1967, McKenzie 1973, Eltayeb 1977, Ahmed & Eltayeb 1978, Rudraiah & Venkatachalappa 1979).

The dissipation parameter β (5b) is unity at the transition layer (14a), and if we chose it as independent variable instead of altitude, the singularities are located at $\beta = 0, 1, \infty$. The simplest special function with three regular singularities is the hypergeometric, and indeed the fields of resistive Alfvèn-gravity waves propagating in an isothermal atmosphere can be expressed exactly in terms of hypergeometric functions of parameters $c = 1$ and:

$$a = 1 + (1 - i) K L \equiv \upsilon, \quad b = 1 - (1 - i) K L \equiv 2 - \upsilon, \quad K \equiv \sqrt{\omega / 2\zeta}, \tag{15a, b, c}$$

where the frequency ω is related quadratically to the effective wavenumber K through (twice) the magnetic diffusivity, $\omega = 2\zeta K^2$. The velocity perturbation spectrum is given by:

$$\begin{gathered} V(z;\omega) = V(0;\omega)\,(ia^2/\omega\zeta + e^{-z/L})^{-\upsilon} \\ \{F(\upsilon, \upsilon - 1; 2\upsilon - 1; (1 - i(\omega\zeta/a^2)\, e^{-z/L})^{-1})\} \\ \{F(\upsilon, \upsilon; 2\upsilon - 1; i\,a^2/\omega\zeta)\}^{-1} \end{gathered} \tag{16}$$

which applies at all altitudes $0 \leq z \leq \infty$, including the transition layer.

Considering standing modes, the spectrum is discrete, and the eigenvalues for the complex frequency ω_n are given by the roots of:

$$0 = F(\upsilon, 2-\upsilon; 1; i\omega\zeta/a^2)$$
$$= 1 + \sum_{p=1}^{\infty} (i\omega\zeta/a^2)^p (p!)^{-2} \prod_{q=0}^{p-1} (q^2 + 2q + 1 + i\omega\zeta/a^2) ; \quad (17)$$

the roots $\omega_n = f_n - i\, d_n$ specify, through their real part, the frequency $f_n = \mathrm{Re}(\omega_n)$, and through minus the imaginary part, the damping rate $d_n = \mathrm{Im}(-\omega_n)$, since $\exp(-i\omega_n t) = \exp(-if_n t) \exp(-d_n t)$. The wave field is the superposition of all standing modes with non-vanishing initial spectrum $V(0;\omega_n)$:

$$v(z,t) = \sum_{f_n, d_n > 0} \mathrm{Im}\{ V(0;\omega_n)\, e^{-if_n t} \}\, e^{-d_n t}$$
$$\{ \partial F(\upsilon, 2-\upsilon; 1; i\omega\zeta/a^2)/\partial\omega_n \}^{-1} F(\upsilon, 2-\upsilon; 1; i(\omega\zeta/a^2)\, e^{-z/L}) , \quad (18)$$

in the case of perfect reflection above the transition layer.

We choose for illustration the case of non-dissipative Alfvèn--gravity waves perfectly reflected from infinity in an isothermal atmosphere, and plot in Fig. 1 the waveforms of the first three standing modes, using an altitude z made dimensionless by dividing by the scale height L. The waveforms are shown for the velocity (top) and magnetic field (bottom) perturbations, made dimensionless by dividing by initial reference values, respectively $V_n, H_n = \{ e^{-i\omega_n t}/\omega_n\, V(0;\omega_n) \} \{ v_n(z,t), (a/B)\, h_n(z,t) \}$ From the waveforms for the first (left), second (centre) and third (right) standing modes it follows that: (i) the velocity perturbation vanishes at the node $z = 0$, whereas the node at infinity (where the density is zero) corresponds to a constant asymptotic amplitude, decreasing in value (when divided by ω_n) and alternating in sign with the order of the mode; (ii) the magnetic field perturbation starts out-of-phase with a non-zero value at $z = 0$, and decays exponentially to zero in the asymptotic regime, which is attained at higher altitude for higher order modes; (iii) as typical of Sturm-Liouville problems, the velocity and magnetic field perturbations of the n-th mode have $(n-1)$ nodes at intermediate altitudes, with interlacing between the nodes of velocity

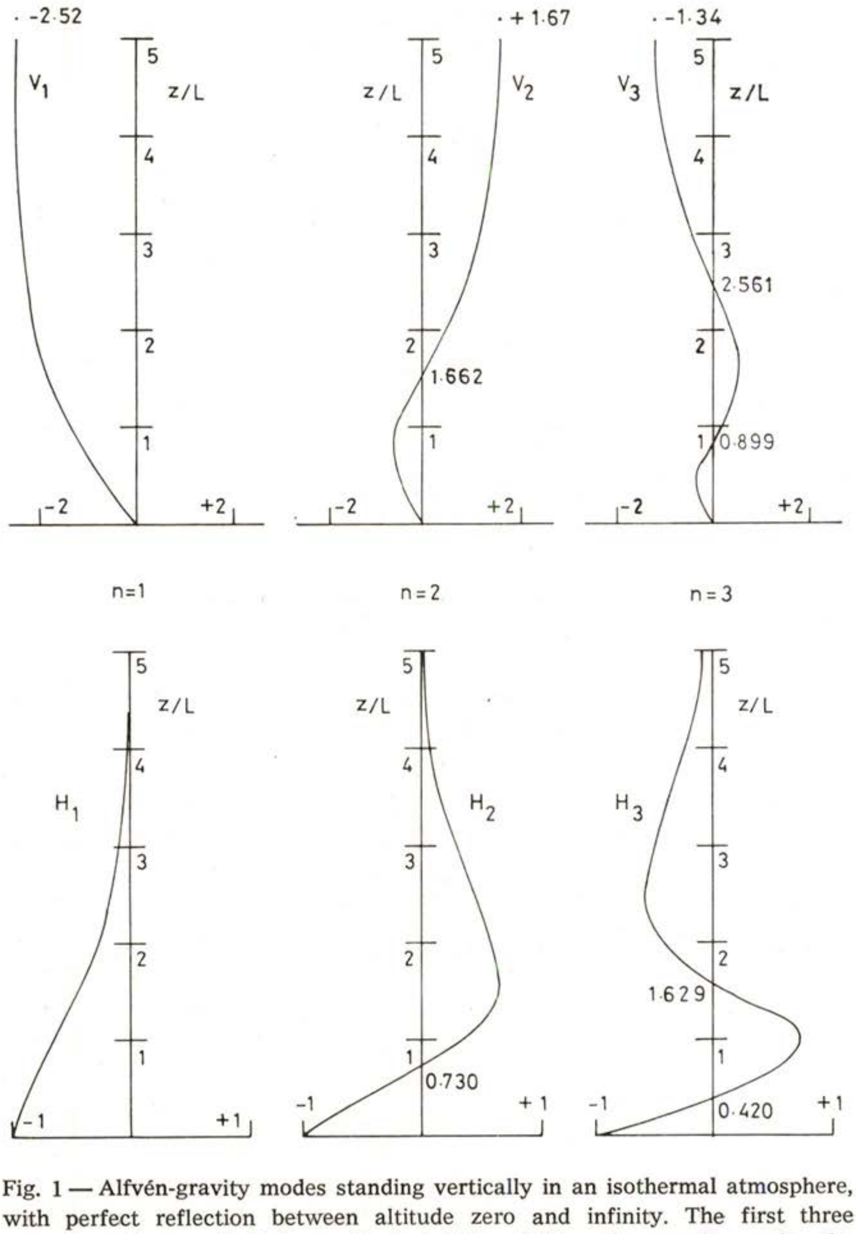

Fig. 1 — Alfvén-gravity modes standing vertically in an isothermal atmosphere, with perfect reflection between altitude zero and infinity. The first three normal modes n = 1, 2, 3 are illustrated by plotting the waveforms for the dimensionless velocity (top) and magnetic field (bottom) perturbations as function of altitude z divided by the scale height L. Note that the magnetic field perturbation starts out-of-phase to the velocity, and decays to zero asymptotically as the velocity tends to a constant value.

and magnetic field of the same mode n, and between sucessive modes $n, n+1$. The compactness (5a, 11a) $\alpha_n = \omega_n L/a = p_n/2 = 1.202, 2.760, 4.327$ respectively for the modes of orders $n = 1, 2, 3$, specifies the local wavelength measured on the scale height $\lambda_n/L = 2\pi/\alpha_n = 5.225,\ 2.276,\ 1.452$, and shows that the atmospheric density change over a wavelength $\Delta \equiv \rho(0)/\rho(\lambda_n) = \exp(\lambda_n/L) = 185.95,\ 9.74,\ 4.27$, is substantial, and thus the effects of atmospheric stratification could not have been neglected or treated approximately by procedures like the W. K. B. J. method.

In Fig. 2 is illustrated the case of non-dissipative Alfvèn--gravity waves propagating vertically, for a compactness (5a) $\alpha = 1$, corresponding to a local wavelength equal to 2π times the scale height $\lambda = 2\pi L$, and an atmospheric density change $\alpha = e^{2\pi} = 535$ in a wavelength. The altitude z is divided by the scale height, and the velocity and magnetic field perturbation spectra for a wave of frequency ω at altitude z are rendered dimensionless by dividing by initial values $\mathsf{V} \equiv V(z;\omega)/V(0;\omega)$ and $\mathsf{H} \equiv (a/B)\ H(z;\omega)/V(0;\omega)$. The amplitude of the velocity perturbation (bottom left) initially grows exponentially on four times the scale height $\sim e^{z/4L}$ (10a), and asymptoticaly grows linearly (6a, 10b); the amplitude of the magnetic field perturbation (bottom centre) evolves differently, since it decays initially on four times the scale height $\sim e^{-z/4L}$, and tends asymptoticaly to a constant value (6b). Since the magnetic field perturbation is horizontal, and hence transverse to the direction of propagation, there is an associated electric current, according to Maxwell's law:

$$\nabla \wedge \mathbf{H} = (4\pi/c)\ \mathbf{J}; \quad j(z,t) = (c/4\pi)\ \partial h(z,t)/\partial z, \qquad \text{(19a, b)}$$

specifies the electric current propagated with the wave, which for a frequency component ω at altitude z is rendered dimensionless by dividing by initial values $\mathsf{J} \equiv 4\pi a^2 J(z;\omega)/c_* B\omega V(0;\omega)$; the magnitude of the electric current (bottom right) decays exponentially with height, initially on four-thirds the scale height $\sim e^{-3z/4L}$, and asymptoticaly on the scale height $\sim e^{-z/L}$, thus tending to zero. The phases (top) of the velocity perturbation (left), magnetic field (centre) and electric current (right), initially vary linearly with altitude, and, as a consequence of the increase of propagation speed with height, the slope increases, so that asymptoticaly the value is finite in all cases.

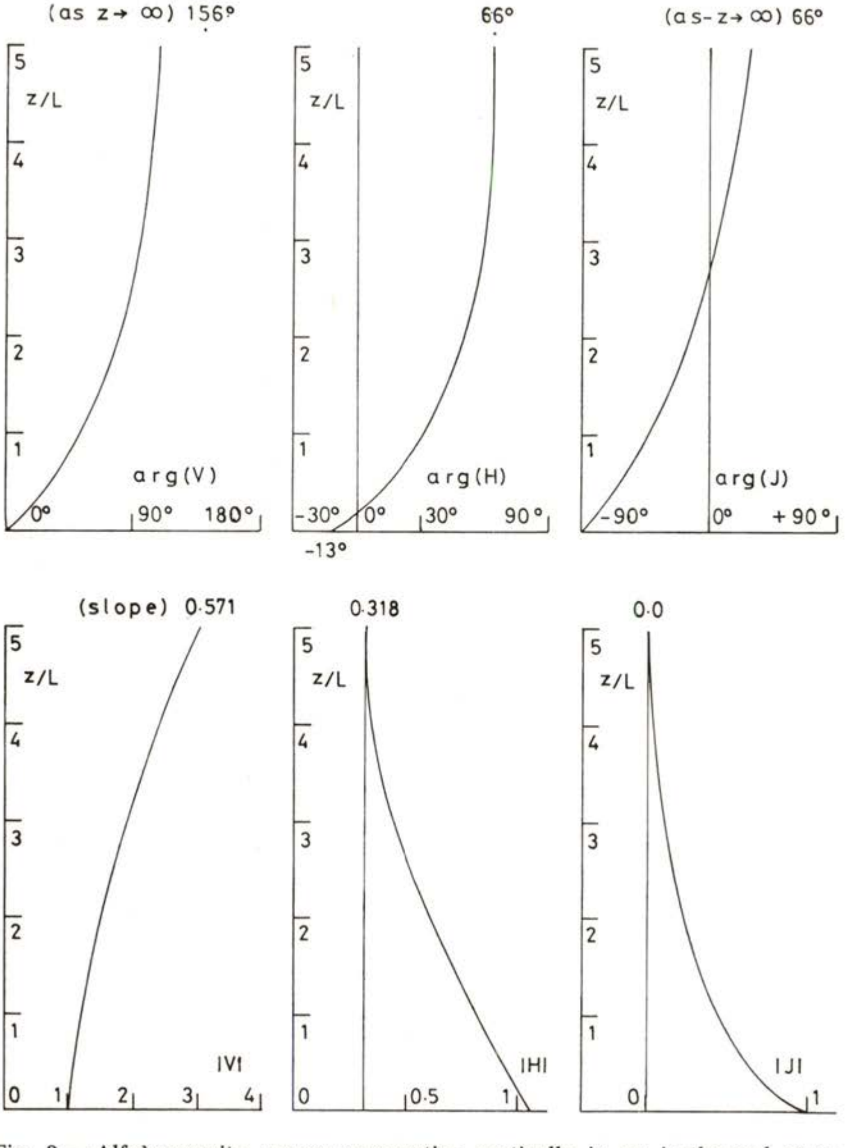

Fig. 2 — Alfvèn-gravity waves propagating vertically in an isothermal atmosphere, for a compacteness parameter unity, corresponding to a change in atmospheric density by a factor of $e^{2\pi} = 535$ over a local wavelength. The wave fields are illustrated by the phases (top) and amplitudes (bottom) of the velocity (left), magnetic field (centre) and electric current (right) propagated with the wave, plotted against dimensionless altitude as in Fig. 1. Note the different initial and asymptotic amplitude laws for each wave variable, and the finite asymptotic phase in all cases.

Since the Alfvèn-gravity waves propagate an electric current j, they supply heat to the atmosphere by Joule effect at the rate $W = (1/\sigma) j^2$, and thus are an effective energy transfer mechanism for atmospheric heating (Osterbrock 1961, Uchida & Kaburaki 1975, Ionson 1982, Campos 1984b). On the other hand, being transversal and incompressible, the Alfvèn modes carry no mass flux. The transfer of mass is more appropriate to the acoustic modes, in particular non-linear or of finite amplitude, which can cause a strong compression of the medium. Acoustic modes can be dissipated by viscosity, which is generally much less effective than electric resistance in transfering energy to the medium. Thus the different modes of magneto-acoustic-gravity waves provide effective mechanisms of transfer of mass, energy and momentum, by propagating from one atmospheric region to another, and then depositing their flux. Having considered Alfvèn-gravity waves, for which the variation of propagation speed with altitude is important (8a), but non linear effects are not, since the Alfvèn number $N \equiv v/A(z) \sim z\, e^{-z/2L}$ decreases with height, we consider next acoustic modes, which will be shown to have contrasting properties.

3 — FILTERING OF HYDRODYNAMIC WAVES AND NON-LINEAR COMPRESSION

Returning to small amplitude magneto-acoustic-gravity waves (eq. (6) of Part I), in an homogeneous medium, the velocity perturbation may be represented by a Fourier decomposition in space and time:

$$v_j(\mathbf{x};t) = \int_{-\infty}^{+\infty} a_j(\mathbf{k},\omega) \exp\{ i(\mathbf{k}.\mathbf{x}-\omega t)\} \, d^3k \, d\omega ; \qquad (20)$$

since application of $(\partial/\partial t, \partial/\partial x_i, \partial/\partial m)$ to v_j is equivalent to multipying a_j by $(-i\omega, i k_i, i(\mathbf{k}\cdot\mathbf{m}))$, where ω denotes the frequency and k_i the wave vector, the homogeneous wave equation $\Box_{ij}\{v_j\} = o$ leads to the algebraic condition $\Pi_{ij}\, a_j = o$, where the dispersion matrix is given by:

$$\begin{aligned}\Pi_{ij} = -\omega^2\, \delta_{ij} + (c^2 + a^2)\, k_i k_j - i\,(k_i g_j - k_j g_i) - i\gamma g_i k_j \\ + a^2\,(\mathbf{k}\cdot\mathbf{m})\,\{(\mathbf{k}\cdot\mathbf{m})\,\delta_{ij} - k_i m_j - k_j m_i\}, \qquad (21)\end{aligned}$$

where c, a are respectively the sound and Alfvèn speeds, and $\mathbf{m} \equiv \mathbf{B}/B$ the unit vector in the direction of the external magnetic field $\mathbf{B}$, assumed constant. In order for waves to exist the amplitude cannot vanish, and the roots of the equation $|\Pi_{ij}| = 0$ specify the dispersion relations $\omega(\mathbf{k})$ for each mode. We can consider three cases: (i) for pure acoustic waves $m_i = o = g_i$, the first two terms of (21) show that the dispersion matrix is quadratic in ω, k, so that the frequency is a linear function of wavenumber, and acoustic waves are isotropic, and non-dispersive, i.e., propagate at the same speed (of sound c) in all directions and for all frequencies; (ii) in the presence of magnetic field $m_i \neq o = g_i$, the dispersion matrix is quadratic on frequency ω and wavevector k_i so that the frequency is linear in the wavenumber k but depends on the wavenormal $n_i = k_i/k$, and magneto-acoustic waves are non-dispersive and anisotropic, i.e., the speed of propagation depends on direction, and the wavefronts are not spherical (they are plane for the Alfvèn mode and curved for the slow and fast modes); (iii) in the presence of gravity $g_i \neq o$, with or without compressibility and/or magnetic field, the dispersion matrix involves terms of the first and second degree, so that the frequency depends non-linearly on the wave vector, and gravity, acoustic--gravity, magneto-gravity, and magneto-acoustic-gravity modes are all anisotropic and dispersive, since the wave speed depends both on direction and frequency.

Similar methods apply to the wave equations in homogeneous elastic solids, either isotropic or crystals (Love 1927, Cady 1946, Schouten 1953, Achenbach 1973, Hudson 1980); surface waves in fluids and solids can also be isotropic or anisotropic, dispersive or non-dispersive, their dispersion relation being determined from the boundary conditions applying at the interface along which propagation occurs. As an example of the derivation of dispersion relations, we consider the determinant of the dispersion matrix (21), in the case of vertical propagation, when the wavevector $\mathbf{k} \equiv (0, 0, k)$ is anti-parallel to gravity $\mathbf{g} \equiv (0, 0, -g)$, taken in the x_3-direction, and the plane (x_1, x_3) can be chosen to contain the direction of the external magnetic field $\mathbf{m} \equiv (m_1, 0, m_3)$:

$$0 = (\omega^2 - a^2 k^2 m_3^2) \begin{vmatrix} \omega^2 - a^2 k^2 m_3^2 & a^2 k^2 m_1 m_3 \\ a^2 k^2 m_1 m_3 & \omega^2 - c^2 k^2 - a^2 k^2 m_1^2 - i\gamma g k \end{vmatrix} \qquad (22)$$

Since (22) is a cubic in ω^2 there are three wave modes, of which one is uncoupled from the other two, as shown by the factor in curved brackets, which corresponds to the dispersion relation $\omega(k)$, phase speed $\mathbf{u}$ and group velocity $\mathbf{w}$:

$$\omega = \pm a k m_3 = \pm a(\mathbf{k}\cdot\mathbf{m}), \quad \mathbf{u} \equiv \omega/k = \pm a(\mathbf{n}\cdot\mathbf{m}),$$
$$\mathbf{w} \equiv \partial\omega/\partial\mathbf{k} = \pm a\,\mathbf{m}; \qquad (23a, b, c)$$

thus the uncoupled mode is an Alfvèn wave (23a), since it propagates energy at the group velocity (23c), which coincides with the Alfvèn speed a along magnetic field lines $\mathbf{m}$, and the wave crests move at the phase speed $u = \mathbf{w}\cdot\mathbf{n}$, which corresponds to the group velocity $\mathbf{w}$ projected on the wavenormal direction $\mathbf{n} \equiv \mathbf{k}/k$.

The other two modes, described by the determinant of (22):

$$\omega^4 - \{(a^2+c^2)\,k^2 + i\gamma gk\}\,\omega^2 + a^2\,(\mathbf{k}\cdot\mathbf{m})^2\,(c^2k^2 + i\gamma gk) = 0, \quad (24)$$

are generally coupled, and are distinguished by their speeds as slow and fast. The slow and fast modes decouple when the sound and Alfvèn speeds are of very different orders of magnitude, i.e., one of the gas or magnetic pressures dominates the other. In hydrodynamics, the magnetic pressure is small compared with the gas pressure $a^2 \ll c^2$, and the equation (24) decouples in two factors:

$$\{\omega^2 - a^2\,(\mathbf{k}\cdot\mathbf{m})^2\}\,\{\omega^2 - c^2k^2 - i\gamma gk\} = 0; \qquad (25a)$$

the first factor specifies again the Alfvèn wave (23a, b, c) propagating along magnetic field lines $\mathbf{m}$ at a small speed a, and is the slow mode, and the second factor specifies an acoustic-gravity wave, which corresponds to the fast mode. In the converse case, of dominant magnetic field $a^2 \gg c^2, \gamma g/k$, in the vertical direction $m_1 = 0, m_3 = 1$, equation (24) factors:

$$\{\omega^2 - a^2k^2\}\,\{\omega^2 - c^2k^2 - i\gamma gk\} = 0, \qquad (25b)$$

into an acoustic-gravity wave (second factor) as in (25a), but now corresponding to the slow mode, whereas the first factor specifies the fast mode:

$$\omega = \pm a k , \; u \equiv \omega/k = \pm a , \; \mathbf{w} \equiv \partial \omega / \partial \mathbf{k} = \pm a \mathbf{n} , \qquad (26a, b, c)$$

which is a wave propagating isotropically in all directions $\mathbf{n}$ at Alfvèn speed a.

Thus we find that the vertical acoustic-gravity wave, of dispersion relation $\omega^2 - c^2k^2 - i\gamma g k = 0$ exists, uncoupled from magnetic modes, in two opposite circunstances: (a) in the case of a magnetic field so weak (25a) that it cannot affect the acoustic-gravity wave; (b) in the case of a magnetic field so strong (25b) that the gas cannot cross magnetic field lines, and the acoustic-gravity wave is forced to propagate along the magnetic flux tube. In the absence of gravity $g = 0$, the dispersion relation reduces to $\omega^2 - c^2k^2 = 0$, implying (26a, b, c) with c replacing a, i.e., isotropic propagation at the sound speed $c^2 = \gamma RT$, where γ is the ratio of specific heats, R the gas constant and T the temperature. The presence of gravity affects acoustic waves by introducing anisotropy, dispersion and filtering, as can be seen from the dispersion relation (second factor in 25a, b) written for the vertical wavenumber k:

$$k^2 + (i/L)\, k - \omega^2/c^2 = 0 , \; L \equiv c^2/\gamma g = RT/g , \qquad (27a, b)$$

which involves the atmospheric density scale height given by (27b). Solving (27a) for the vertical wavenumber:

$$k = -(i/2L)\, \{ 1 \pm \sqrt{1 - \omega^2/\omega_*^2} \} , \qquad (28a)$$

$$\omega_* \equiv c/2L = (g/2)\sqrt{\gamma/RT} , \qquad (28b)$$

we conclude that it is pure imaginary $k = -i\delta/2L$, and the wave field non-oscillating $\exp(ikz) = \exp(\delta z/2L)$, for frequencies below (28b) $\omega < \omega_*$, which is designated cut-off frequency, since below it propagating waves cannot exist. Above the cut-off frequency $\omega > \omega_*$, the wavenumber k (28a) has a real part:

$$k = -i/2L \pm K , \; K \equiv (\omega/c)\sqrt{1 - \omega_*^2/\omega^2} , \qquad (29a, b)$$

and propagation is possible, so that the cut-off frequency ω_* (28b) separates the standing ($\omega < \omega_*$) and propagating ($\omega > \omega_*$) parts of the spectrum, and specifies the properties of the atmosphere as a filter of acoustic waves.

The quantity K defined by (29b) plays the role of an effective wavenumber, since by (29a):

$$\exp i (kz - \omega t) = \exp (z/2L) \cdot \exp i (Kz - \omega t) \qquad (30)$$

it determines the phase $Kz - \omega t$ of a propagating wave $\omega > \omega_*$. The effective wavenumber K (29b) simplifies to the acoustic form $K \sim \omega/c$ for frequencies much higher than the cut-off $\omega^2 >> \omega_*^2$, is smaller than the acoustic value $K < \omega/c$ for intermediate frequencies $\omega > \omega_*$, and vanishes $K = 0$ at the cut-off $\omega = \omega_*$ when propagation becomes impossible. The phase speed u and group velocity w corresponding to the effective wavenumber K (29b):

$$u \equiv \omega/K = c \left\{ 1 - \omega_*^2/\omega^2 \right\}^{-1/2}, \qquad (31a)$$

$$w \equiv \partial \omega / \partial K = c \left\{ 1 - \omega_*^2/\omega^2 \right\}^{1/2}; \qquad (31b)$$

coincide with the sound speed $u \sim c \sim w$ only at high-frequencies $\omega^2 >> \omega_*^2$; at intermediate frequencies $\omega > \omega_*$ the phase speed is higher than the sound speed and the group velocity lower $u > c > w$. At the cut-off $\omega = \omega_*$ the phase speed diverges $u \rightarrow \infty$ since the wave ceases to oscillate and wavecrests disappear, and the group velocity vanishes $w \rightarrow 0$ since the energy flux vanishes for standing modes. The factor $\exp (z/2L)$ shows that the amplitude of acoustic waves grows exponentially on twice the density scale height in an isothermal atmosphere, and thus, even if the initial amplitude is small, after propagating a few scale heights the waves grow to finite amplitude, so that non-linear effects become important and shock formation may occur. This law of amplitude growth can also be obtained from energy conservation, bearing in mind that the sound speed is constant in an isothermal atmosphere: (i) the equipartition of kinetic and compression energies at all altitudes implies that the total energy $E = \rho v^2$ is twice the kinetic part, and the energy flux is $F = Ec = \rho v^2 c$; (ii) since the energy flux must be conserved for a non-dissipative wave $v \sim \rho^{-1/2}$, and as the mass density decays exponentially on the scale height $\rho \sim \exp(-z/L)$ the wave velocity perturbation grows exponentially on twice that scale $v \sim \exp(z/2L)$.

There is a considerable contrast between the properties of Alfvèn-gravity (§2) and acoustic-gravity (§3) waves, say, in an isothermal atmosphere: (i) the latter have a constant propagation speed, namely the sound speed c, whereas the former are accelerated with height at an exponentially increasing Alfvèn speed A (8a); (ii) the velocity perturbation increases linearly with altitude $v \sim z$ in the former (6a) and exponentially $v \sim e^{z/2L}$ in the latter (30) cases; (iii) for the former, the Alfvèn number $N \equiv v/A \sim z\ e^{-z/2L}$ reduces with altitude, whereas for the latter the Mach number $M \equiv v/c \sim e^{z/2L}$ increases with height; (iv) for the former a linear, non-homogeneous theory is appropriate, whereas for the latter a non-linear, homogeneous approach may be needed; (v) the latter satisfy equipartition of (kinetic and compression) energies at all altitudes, and the former violates the initial equipartition of (kinetic and magnetic) energies as it propagates upward. These contrasts for non-dissipative waves (Campos 1983a, b) extend to the dissipative modes (Campos 1983c, d), since: (vi) among the main dissipation mechanisms, viscous damping is effective on hydrodynamic and resistive damping on hydromagnetic waves; (vii) the electrical resistance depends mainly on temperature and is approximately constant in an isothermal atmosphere, whereas the kinematic viscosity, defined as the ratio of the static viscosity to the mass density increases with height as the atmosphere becomes more rarefied; (viii) resistive dissipation is important at low altitudes for Alfvèn waves, when their propagation speed is small, and becomes negligible at high-altitudes, whereas for acoustic modes viscous damping increases in importance as the kinematic viscosity grows with height, and is most significant at high altitudes.

The simple preceding example of the high-pass filtering of vertical acoustic-gravity waves in a magnetic field tube illustrates three general properties: (a) the filtering of waves in stratified media, e.g., three-dimensional acoustic-gravity waves have *two* cut-off frequencies (one vanishes in the case of vertical propagation outlined before), in atmospheres either nearly isothermal (Lamb 1879, Moore & Spiegel 1964, Yih 1965, Yeh & Liu 1974, Campos 1984a) or with strong temperature gradients (Lamb 1910, Groen

1948, Thorpe 1968, Lindzen 1970, Campos 1983a); (b) the propagation along tubes is also relevant in the acoustics of ducts of variable cross-section, either without a mean flow, such as the horns of loudspeakers and musical instruments (Webster 1919, Ballantine 1927, McLachlan 1934, Jordan 1963, Olson 1972, Zamorski & Myrzykowski 1981, Kergomard 1982, Bostrom 1983, Campos 1984c), or with a mean flow, such as the nozzles of jet engines (Powell 1959, Eisenberg & Kao 1969, Morfey 1971, Nayfeh, Kaiser & Telionis 1975, Nayfeh, Shaker & Kaiser 1980, Campos 1978, 1984d); (c) the growth in wave amplitude with altitude leads to non-linear waves and shock formation (Riemann 1860, Raiser & Zeldovitch 1966, Wentzel & Solinger 1967, Chiu 1971, Whitham 1974, Roberts & Rae 1982, Campos 1984e), which is oppposed and delayed by dissipative effects (Lighthill 1951, 1978; Yanowitch 1967a, b, 1969; Lyons & Yanowitch 1974; Campos 1983c, d). In these as in all other types of waves, it is possible to consider three types of propagation and dissipation theories: (I) for waves of small amplitude in media for which the propagation speed and damping rates are uniform, the wave equations are linear with constant coefficients, and can be solved by Fourier analysis, leading to the use of dispersion relations, as in the preceding example (§3) of acoustic-gravity waves in an isothermal atmosphere; (II) for waves of small amplitude in an inhomogeneous or strongly stratified medium, for which the speed of propagation and dissipation parameters are non-uniform, the wave equations are linear with variable coefficients, dispersion techniques are inadequate, and exact solutions must be obtained, often in terms of special functions, as illustrated in the case of resistive Alfvèn-gravity waves (§2); (III) for waves of large amplitude, which cause a disturbance in the medium of propagation comparable with the mean state, the wave equations are non-linear and the methods of superposition generally do not apply, and apart from cases of short wavelength where non-linear ray theory may be used, the derivation of exact solutions may depend on finding a special transformation into a known equation, as in the example which follows in the remaining part of the present section.

We consider as example a one-dimensional acoustic wave of finite amplitude, with dissipation by the kinematic viscosity η, for which the velocity v and density ρ are functions only of position z and time t, and satisfy the exact, non-linear equations of continuity and momentum:

$$\partial \rho / \partial t + v \, \partial \rho / \partial z + \rho \, \partial v / \partial z = 0 \tag{32a}$$

$$\partial v / \partial t + v \, \partial v / \partial z + (c^2 / \rho) \, \partial \rho / \partial z = \eta \, \partial^2 v / \partial z^2 ; \tag{32b}$$

in the latter we have substituted the pressure gradient $(\partial p / \partial z) = c^2 (\partial \rho / \partial z)$, where $c^2 = (\partial p / \partial \rho)_s$ is the adiabatic sound speed. Multiplying (32a) by $\pm c/\rho$ and adding to (32b), we obtain:

$$\{ \partial / \partial t + (v \pm c) \; \partial / \partial z \} \; J_\pm = \eta \; \partial^2 v / \partial z^2 , \tag{33a}$$

$$J_\pm \equiv v \pm \int (c/\rho) \; d\rho , \tag{33b}$$

which can be interpreted as follows: (i) in the absence of viscosity $\eta = 0$ and for linear waves $v << c$, from $J_+ \sim 2v$ and $\partial v / \partial t + c \; \partial v / \partial z = 0$, it follows that the velocity perturbation $v(z, t) = f(z - ct)$ propagates at sound speed without deformation; (ii) in the non-linear case the Riemann invariants $J_\pm$ are conserved along the characteristics $\Gamma_\pm$, which are the 'trajectories' of the wave travelling at a speed $(dx/dt)_\pm = v \pm c$ equal to the superposition of the velocity v and sound speed $\pm c$, in the same or opposite direction; (iii) in the presence of viscosity $\eta \neq 0$ the Riemann invariants $J_\pm$ (33b) decay along the characteristics $\Gamma_\pm$ since (33a) may be viewed as non-linear diffusion equations, and the wave profile is deformed by two opposing effects: steepening by non-linear convection and decay by viscous dissipation.

Considering a simple wave, for which the invariant J_- is zero, the invariant J_+ can be expressed in terms of the propagation speed $u \equiv v + c$, so that we obtain Burger's equation:

$$\partial u / \partial t + u \, \partial u / \partial z = (\eta / 2) \, \partial^2 u / \partial z^2 ; \tag{34}$$

the latter balances the linear local and non-linear convective acceleration against dissipation by the kinematic viscosity halved, and apart from the factor 1/2 coincides with the one-dimensional form of Navier-Stokes equation. The equation (34) can be transformed, by means of the change of variable (Burgers 1948, Cole 1951, Hopf 1951):

$$\Phi (z , t) = \exp \{ - \eta^{-1} \int_{-\infty}^{z} u (y , t) \, dy \} , \qquad (35a)$$

$$u (z , t) = - \eta \; \partial (\log \Phi) / \partial z , \qquad (35b)$$

into the linear heat equation:

$$\partial \Phi / \partial t = (\eta / 2) \; \partial^2 \Phi / \partial z^2 , \qquad (35c)$$

of which many solutions are known (Fourier 1876, Carslaw & Jaeger 1946).

As an example of the use of the transformation (35a), we consider a velocity pulse of magnitude U emitted at altitude $z = 0$:

$$u (z , o) = U \, \delta (z) , \Phi (z , o) = H (- z) + e^{-U/\eta} \; H (z) , \quad (36a, b)$$

using the properties of Dirac's delta δ and Heaviside's unit H functions (Schwartz 1949, Lighthill 1958); the solution of the classical (Fourier 1876, Carlslaw & Jaeger 1946) heat problem (35c, 36b) specifies $\Phi (z , t)$ from which we may calculate $u (z , t)$ using (35b), viz.:

$$u(z,t) = (2\eta / \pi t)^{1/2} \; \{ \exp(- z^2 / 2\eta t) / \mathrm{erfc} \; (z / \sqrt{(2\eta t)}) \} ; \qquad (37a)$$

the pulse thus broadens with time to a scale $\chi \sim \sqrt{(2\eta t)}$, and asymptoticaly $z \rightarrow \infty$ as the error function $\mathrm{erfc} \rightarrow 1$, it takes a pure Gaussian shape:

$$u \sim (2\eta / \pi t)^{1/2} \exp (- z^2 / 2\eta t) \sim (2\eta U / \pi z)^{1/2} \exp (- Uz / 2\eta) , \; (37b)$$

where in the second expression we have used $z / t \sim U$.

The dissipation by viscosity acts as a heat source (per unit volume):

$$\begin{aligned} Q (z) &= \rho \, \eta \, (\partial v / \partial z)^2 = \{ 2 \, \rho \, \eta / (\gamma + 1)^2 \} \; (\partial u / \partial z)^2 \\ &\sim \{ \rho \, U^3 / \pi \, (\gamma + 1)^2 z \} \; \exp (- Uz / \eta) , \end{aligned} \qquad (38a)$$

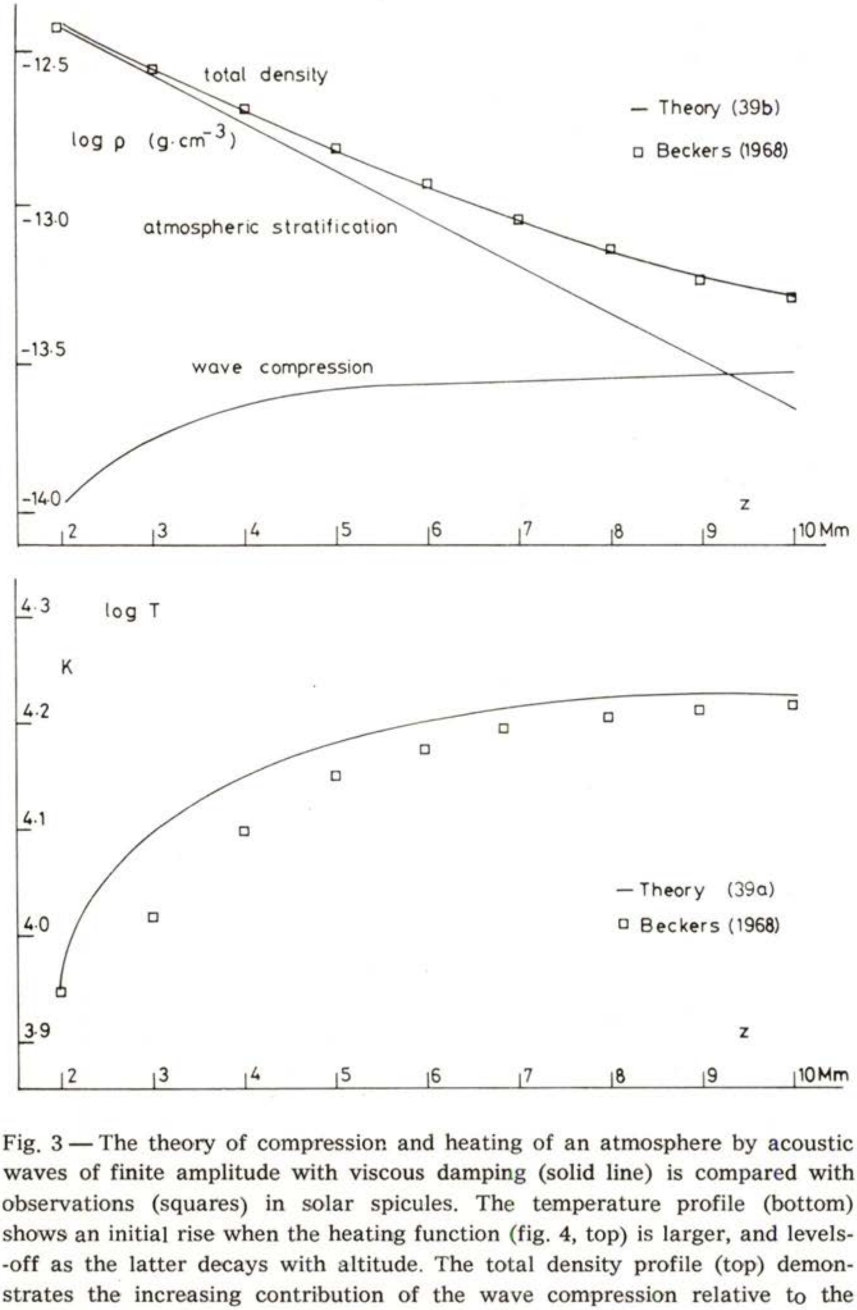

Fig. 3 — The theory of compression and heating of an atmosphere by acoustic waves of finite amplitude with viscous damping (solid line) is compared with observations (squares) in solar spicules. The temperature profile (bottom) shows an initial rise when the heating function (fig. 4, top) is larger, and levels--off as the latter decays with altitude. The total density profile (top) demonstrates the increasing contribution of the wave compression relative to the atmospheric stratification, showing that the presence of waves is essential to explain the observed total density.

which in a rarefied medium is balanced by the thermal radiation flux:

$$Q = dS/dz \,, \; S = (16 \nu / 3 \varepsilon) \; T^3 \; dT/dz \,, \qquad (38b, c)$$

which is proportional to the cube of temperature and temperature gradient through a factor involving Stefan-Boltzmann constant ν and the opacity ε. Integrating (38b) with (38a, c) we obtain the temperature profile of a medium in radiative equilibrium, heated by viscous dissipation of acoustic waves of finite amplitude:

$$\{ T(z) \}^4 = \{ T(z_0) \}^4 + (B/A) \; \{ E_2(A z_0) - E_2(A z) \} \,, \qquad (39a)$$

where E_2 is the exponential integral of order 2 and $A \equiv U/\eta$, $B \equiv 3 \rho U^3 \varepsilon / 4 \nu \pi (\gamma + 1)^2$. The medium may be an atmosphere, provided that the stratification be gradual on the scale of the wave, and in this case the total mass density consists of: (i) the atmospheric stratification which decays exponentially on a scale D from its initial value ρ_1 at altitude z_0 ; (ii) a wave compression which increases from the initial value ρ_2 at altitude z_0 according to an adiabatic relation with temperature:

$$\rho(z) = \rho_1 \; \exp \{ - (z - z_0) / D \} + \rho_2 \; \{ T(z) / T(z_0) \}^{\gamma / (\gamma - 1)} \,. \qquad (39b)$$

The temperature and density profiles (39a, b) are plotted versus altitude in Fig. 3, showing satisfactory agreement with the observations (Beckers 1968, 1972) in spicules, which are regions of the sun (Bray & Loughhead 1974, Athay 1976) where matter moves upward towards the corona, compensating for the mass loss due to the solar wind. It can be seen that: (bottom) the temperature gradient reduces with altitude as viscous dissipation becomes weaker and thermal radiation more effective; (top) the wave compression gives an increasing contribution to the total density compared with the atmospheric stratification. The presence of the compression front due to the acoustic-gravity wave is thus essential to the adequate modelling of the density profile. The mass and energy transfer to the atmosphere can be related to the

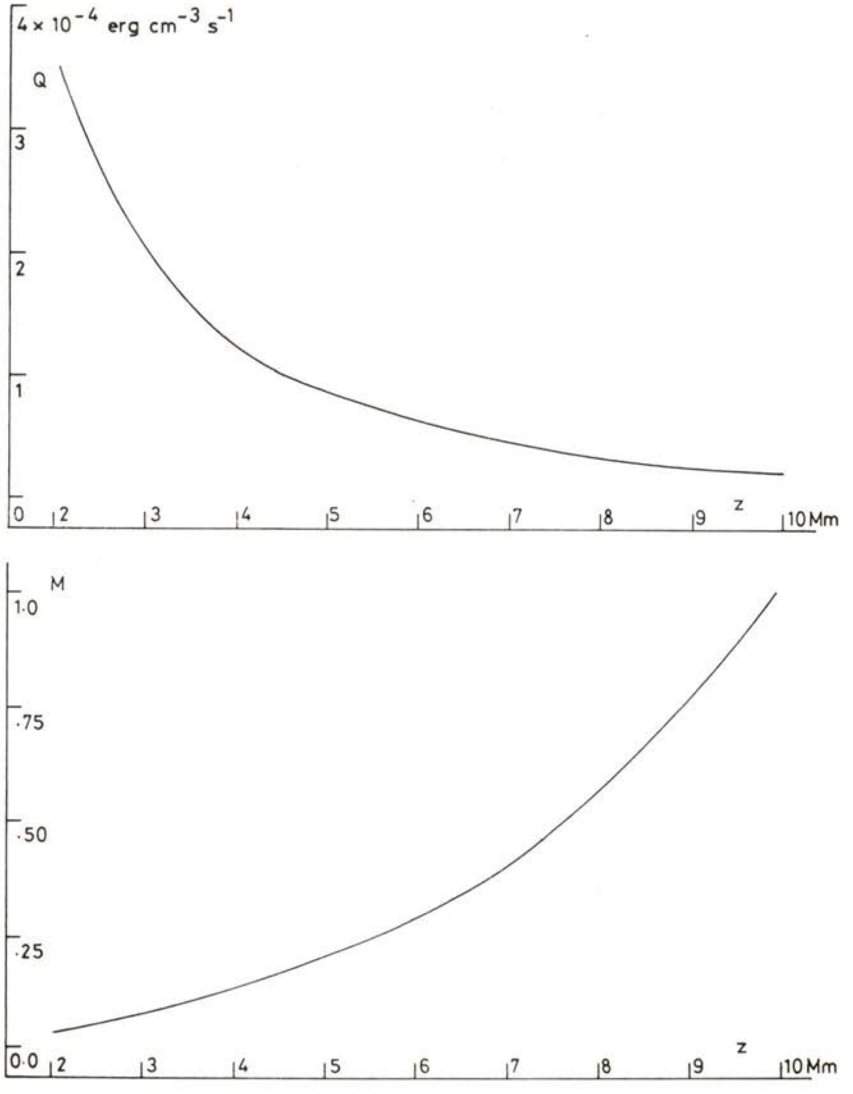

Fig. 4 — An acoustic wave tends to steepen its wavefront and form a shock due to non-linear effects, and this tendency is opposed by viscous dissipation, which extracts energy from the wave and transfers it to the medium in the form of heat. The heating function (top) specifying the rate of transfer of energy from the acoustic waves to the solar atmosphere decays with altitude, as there is less energy per unit volume left in the wave; in spite of this energy loss, the decaying atmospheric density implies that the wave compression becomes gradually more significant in relative terms, and the Mach number (bottom) increases steadily up to shock strength.

properties of the wave, shown in Fig. 4: (top) the heating function due to viscous dissipation decays with altitude, as the temperature levels-off (Fig. 3, bottom); (bottom) the Mach number $M = v/c$ increases rapidly with altitude, as the wave compression becomes more significant compared with the atmospheric density (figure 3, top), showing the growth of the wave from small amplitude $M^2 << 1$ at low altitude to shock strength $M \sim 1$ at the top.

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