A COMPENDIUM ON DIRECTED AND 3-D UNDIRECTED LATTICE DATA

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ABSTRACT — Lattice data on configurational histograms are given for three dimensional undirected bond (site) clusters according to cycle discriminations and for directed lattice animals with both perimeter and cycle discriminations.

INTRODUCTION

Configurational studies have remained one of the foundations of critical phenomena, ever since their study began, despite strong competition from transfer-matrix methods and all the various types of calculations spawned by the renormalization group theory. In this presentation we are concerned with the statistics of connected clusters relevant to the percolation and animal problems, as covered in a previous compendium of data [1]. The present summary lists results pertaining to both the normal (i.e. undirected) and directed problems, and aims to complete the previous illustration in the light of both the current knowledge and the significant theoretical advances that have occurred in the intervening three years. In the domain of normal percolation and normal lattice animals these are non-existent (but see [5]). However, for directed percolation and the relevant animals exact results now include the dominant and sub-dominant singularities for dimensions 2 and 3, their connection in all dimensions to the value of the Yang-Lee edge singularity [2], [3], as well as some multiplicities for the most significant lattices in 2 and 3 dimensions [3].

In this paper, the normal models are only listed in 3 dimensions and we have run as close a parallel as possible with the earlier presentation. The data are therefore divided into 4 groups: cyclomatic number distributions (in normal percolation), fixed size cycle groupings (normal animals), fixed size directed percolation groupings and cyclomatic number distributions (directed animals). The notation conforms to the one applied throughout the previous paper, so that

s – denotes the number of cluster sites
b – denotes the number of cluster bonds
$c = b - s + 1$ denotes the cyclomatic number of a connected cluster
e – denotes the external bond ("energy") perimeter
t – denotes the perimeter in the percolation sense

A — *Cyclomatic number distributions in percolation*

In 3 dimensions the weighting of configurations by its cyclomatic number continues to be of interest. In normal percolation the weighted Euler's law acts as a sum rule for configuration derivations that include the three indices *b*, *s*, and *t*. In fact from the expansions of the moments of the cluster size distribution for bond

$$< s^k > = \sum_{s,b,t} s^k g_{sbt} p^b (1-p)^t \qquad \text{A1}$$

and site percolation

$$< b^k > = \sum_{s,b,t} b^k g_{sbt} p^s (1-p)^t \qquad \text{A2}$$

one has for $k = 1$

$$< s^1 > = 1 - (1-p)^z \qquad \text{A3}$$

and

$$< b^1 > = 1/2\ z\ p^2 \qquad \text{A4}$$

with *z* the coordination number of the lattice.

Our results are for the sets of histograms $\Sigma_b\ b\ g_{sbt}$ for the simple cubic, body-centred cubic and face-centred cubic site animals, and for $\Sigma_s\ s\ g_{sbt}$ on the diamond, simple cubic, and face-centred cubic lattices. For the first set we have managed to complete valence discriminations on all three lattices (a particularly lengthy task for the face-centred cubic lattice). From these the number of bonds in a cluster follows through the laws

$$\sum_{v} s_v = s \qquad \text{A5}$$

$$\sum_{v} v\, s_v = 2\,b \qquad \text{A6}$$

where s_v is the number of sites with alence ν and the summations run from 1 to z. Rather than pursue the same line for bond percolation we have partitioned the data in [4] according to the number of sites in each cluster and weighted them accordingly. Equation A3 provides, as usual, a consistency check on such manipulations. The task is very easy. On the lower coordination number side it can be supplemented, if required, with analyses of those few space types whose contribution to the added perimeter through the yield factor technique stretches long enough to involve an overlap with the contribution of strongly embedded clusters of the following cyclomatic number. The cardinal rules of this derivation are stated in [1], section D.

B — *Fixed size energy groupings (normal animals)*

The most relevant result applicable is ref. [5], which throws light on the structure of the dominant singularities for fixed cycle animals. We complete the simple cubic results of ref. [5] with the rest of the possible cycle values and add the diamond site results for animals.

Since our previous comments in [1] were written in the light of the then current ideas, that basically relied on a logmultiplicity for the histogram that would ultimately be linear in the cluster size, it is now clear from [5] that only the prefactor and the exponent can make the shape of these histograms evolve (the multiplicity associated with each cycle value is constant and equal to the tree multiplicity). As could be expected, the diamond lattice gives no more than a rather faint support to this rule.

C — *Fixed size directed percolation groupings*

Directed lattice animals have greatly benefited from the attention of Deepak Dhar and his collaborators [2], [3], and K. De'Bell has derived unpublished 3 and 2 dimensional perimeter polynomials as a basis for his calculations of the usual critical exponents in directed percolation [6], [7]. In this section we list results on the simple quadratic, triangular, simple cubic and hypercubic 4-dimensional lattices (site problem). These typically add two to three more terms to the susceptibility-like exponent series, although further efforts are necessary for a significant refining of the p_c estimates and the γ values in refs. [6] and [7].

These susceptibility series provide consistency checks on the present data, while for the total number of clusters [2] and [3] furnish further numbers, on the totals of lattice animals with a given size.

Ref. [3] is particularly interesting, since a good alternative derivation relies on the use of compact source clusters. Although a recursion relation with the generality of that in ref. [3] valid for the total number of clusters has not been proposed, the two index discriminations required for the perimeter polynomials can be written through inspection. Putting $g^{(i)}_{s,t}$ as the total number of animals from a compact source cluster with length i, and using the simple quadratic lattice

$$g_{s\,t} = 2\ g_{s-1,\,t-1} + g^{(2)}_{s\,t} \qquad \text{C1}$$

$$g^{(2)}_{s\,t} = 3\ g_{s-3,\,t-2} + g^{(3)}_{s+1,\,t-1} + g^{(3)}_{s\,t} + 2\ g_{s-2,\,t-1} \qquad \text{C2}$$

$$\sum_{s,\,t} g^{(i)}_{s\,t}\, p^{s}\, (1-p)^{t} = p^{\,\sum_1^i m} \qquad \text{C3}$$

This last equation is very useful. For $i = 1$, it is no more than the sum rule for the primary species in directed percolation. For higher values of i additional sets of perimeter polynomials can be used to either check or substitute the lengthier complete polynomials. The configurational work is therefore lessened while parallel series for the moments of the cluster size distribution can be obtained by the formula

$$< s^k > = \sum_{s,\,t} s^k g^{(i)}_{s\,t} p^s (1-p)^t \qquad \text{C4}$$

where equation C3 is implicity contained for $k = 0$, $k = 1$ leads to susceptibility series (for the exponent γ) and the sum rule C3 provides further coefficients.

D — *Cyclomatic number distributions (directed animals)*

There is no available information on cycle discrimination for directed lattice animals and critical properties have only one significant point of reference: the result on the correlation exponent for two-dimensional trees obtained by Nadal et al. [8], by the transfer-matrix technique to a high degree of precision. For our studies on the cyclomatic structure of directed animals we have combined straightforward counting, compact source generation and valence discrimination. Note that unlike the undirected models, in the present instance, the bond expectancy rule for site percolation and the site expectancy rule for bond percolation cannot be used. There are no closure sum rules that conveniently test the overall consistency of the discriminations. Unlike earlier valence studies there are 3 possible options in directed models: incoming valence, outgoing valence and total valence. We have made extensive studies — not listed here — on outgoing valence and in terms of these the linkage rule for cyclomatic number calculations is

$$\sum_{v=0}^{z/2} v\, s_v = b \qquad \text{D1}$$

We have used outgoing valence studies on all the simple quadratic site problem histograms. For the simple cubic site problem only the last two terms have not been checked in this way. The data are here presented in a combination of the various references presented: thus, the generation of complete bond discriminations from compact sources of length 2 on the simple quadratic lattice is obtained by the law

$$g_{s\,b} = g^{(2)}_{s\,b} + 2\ g_{s-1,\,b-1} \qquad \text{D2}$$

and other linkage rules can be related in a similar manner (for example, on the simple cubic, they will involve the embedding of compact sources $g^{(2)}_{s\,b}$, expanding three-dimensionally, and of $g^{(3)}_{s\,b}$).

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APPENDIX

CYCLOMATIC NUMBER DISTRIBUTION IN PERCOLATION

A — *Simple cubic site problem*

	s = 2	$\Sigma_b\ bg_{sbt}$	25		240
10		3	26		15
	s = 3			s = 7	
13		24	18		6
14		6	21		480
	s = 4		22		4500
15		24	23		16440
16		156	24		31488
17		72	25		39816
18		9	26		34404
	s = 5		27		15408
17		48	28		3240
18		420	29		360
19		936	30		18
20		624		s = 8	
21		144	21		96
22		12	22		42
	s = 6		23		2304
18		30	24		17196
20		1536	25		65904
21		3864	26		154050
22		5808	27		245040
23		4824	28		284028
24		1620	29		245676

30	129660
31	36816
32	5712
33	504
34	21
s = 9	
23	456
24	852
25	12144
26	71448
27	283704
28	706248
29	1332984
30	1902468
31	2069100
32	1770576
33	1033824
34	362904
35	74520
36	9216
37	672
38	24
s = 10	
25	2352
26	8181
27	58224
28	352020
29	1232616
30	3303642
31	6821196
32	11207616
33	14634960
34	15332598
35	12894384
36	8024256
37	3284916
38	842694
39	136176
40	13932
41	864
42	27

A — *Body-centred cubic site problem*

s = 2	$\Sigma_b \, bg_{sbt}$
14	4
s = 3	
17	24
19	24
20	8
s = 4	
20	132
22	240
23	96
24	108
25	72
26	12
s = 5	
21	24
23	680
24	120
25	1752
26	752
27	1560
28	1344
29	576
30	432
31	144
32	16
s = 6	
24	288
25	120
26	3736
27	1200
28	11544
29	7144
30	14436
31	13128
32	10684
33	10104
34	5220
35	2400
36	1080
37	240
38	20
s = 7	
25	72
26	96
27	2472
28	1080

29		22608
30		14664
31		70968
32		58032
33		119376
34		115872
35		129744
36		127224
37		90912
38		67320
39		40560
40		16920
41		6888
42		2160
43		360
44		24
	s = 8	
26		56
28		924
29		864
30		21288
31		17328
32		139260
33		129960
34		457020
35		455128
36		926712
37		997656
38		1259788
39		1303104
40		1183764
41		1049648
42		757584
43		472896
44		282244
45		129048
46		47460
47		15792
48		3780
49		504
50		28
	s = 9	
26		8
29		504
31		14280
32		15400
33		163152
34		191016
35		909392
36		1058856
37		3093864
38		3628232
39		6992304
40		8188248
41		11147440
42		12390048
43		12844488
44		12532480
45		10577016
46		8330232
47		5920960
48		3536496
49		1925496
50		930536
51		354264
52		113568
53		31296
54		6048
55		672
56		32

A — *Face-centred cubic site problem*

	s = 2	$\Sigma_b \, bg_{sbt}$
18		6
	s = 3	
22		24
23		24
24		60
	s = 4	
24		12
26		132
27		192
28		360
29		432
30		474
	s = 5	
28		192
29		48
30		888
31		1560

32	2340
33	3840
34	5352
35	4848
36	3384
s = 6	
30	66
31	264
32	1452
33	1512
34	7938
35	10152
36	19608
37	33792
38	44460
39	58896
40	60828
41	45840
42	23310
s = 7	
33	504
34	1056
35	3696
36	14448
37	21672
38	56400
39	99312
40	173712
41	264216
42	427296
43	567432
44	658944
45	732672
46	619608
47	394200
48	157092
s = 8	
35	408
36	1008
37	9192
38	16560
39	48504
40	139044
41	227424
42	523392
43	894624
44	1529868
45	2518632
46	3795504
47	5293128
48	7015092
49	8158740
50	8553690
51	8040408
52	5828280
53	3198216
54	1049538
s = 9	
37	120
38	3096
39	9624
40	30264
41	117120
42	238488
43	585000
44	1324284
45	2472480
46	4922808
47	8433096
48	14548080
49	23360916
50	35586816
51	51293712
52	68905152
53	86741136
54	101184072
55	105531120
56	99223488
57	81340632
58	51984224
59	24993376
60	6972840

A — *Diamond bond problem*

	$\Sigma_s \, sg_{sbt}$
b = 1	
6	4
b = 2	
8	18
b = 3	
10	88
b = 4	
12	455

b = 5	
13	72
14	2376
b = 6	
12	12
15	1008
16	12474
b = 7	
14	168
17	9984
18	65488
b = 8	
16	1656
18	1728
19	82080
20	343791
b = 9	
17	540
18	13572
19	1280
20	33180
21	605040
22	1805440
b = 10	
16	54
18	600
19	10380
20	99780
21	22528
22	416196
23	4152720
24	9489062
b = 11	
17	120
18	1020
20	10560
21	130152
22	683232
23	319968
24	4187592
25	27118656
26	49936536
b = 12	
16	10
19	2112
20	12606
22	148608
23	1309032
24	4556844
25	3828032
26	36671700
27	170927328
28	263195972

A — *Simple cubic bond problem*

b = 1	$\Sigma_s \, sg_{sbt}$
10	6
b = 2	
14	45
b = 3	
17	48
18	332
b = 4	
16	12
21	960
22	2430
b = 5	
20	240
24	1620
25	11952
26	17802
b = 6	
23	756
24	2976
27	2800
28	39900
29	123312
30	131236
b = 7	
22	108
26	1848
27	18732
28	30576
29	3072
31	128832
32	591216
33	1166976
34	973800

	b = 8
25	504
26	2646
28	3264
30	86112
31	278256
32	288216
33	82944
34	263520
35	2695248
36	7069032
37	10558944
38	7266429
	b = 9
24	56
27	1696
29	23136
30	38832
32	88128
33	226908
34	1815912
35	3330288
36	2928828
37	1677600
38	9708450
39	39865920
40	76089120
41	92969640
42	54472030

A — *Face-centred cubic bond problem*

b = 1	$\Sigma_s \, sg_{sbt}$
22	12
	b = 2
31	72
32	126
	b = 3
30	24
39	128
40	768
41	1488
42	1304
	b = 4
38	120
39	480
40	492
48	6150
49	9480
50	20880
51	22920
52	13695
	b = 5
37	48
38	96
47	7320
48	8640
49	13080
50	7560
55	2304
56	37008
57	103536
58	215280
59	286704
60	393300
61	321840
62	145404
	b = 6
36	8
46	4080
47	3240
48	2520
54	4440
55	62928
56	150480
57	255600
58	261216
59	247788
60	105888
63	94192
64	237720
65	1172304
66	2373616
67	3740352
68	5592804
69	6170472
70	6544944
71	4284000
72	1557962

b = 7		68	5663280
45	1200	69	4138092
46	480	70	1406328
53	4464	71	1619520
54	54288	72	3240960
55	106416	73	13946112
56	138528	74	27036480
57	94752	75	52254336
58	46620	76	75166848
62	227976	77	100550400
63	523824	78	119089344
64	2284800	79	116656896
65	4064928	80	100660752
66	5450424	81	55223040
67	6644232	82	16817664

FIXED SIZE ENERGY GROUPINGS (NORMAL ANIMALS)

B — *Simple cubic bond animals*

b = 1	g_{sb}	b = 7	
2	3	6	18
b = 2		7	7308
3	15	8	357987
b = 3		b = 8	
4	95	7	450
b = 4		8	81981
4	3	9	3104013
5	678	b = 9	
b = 5		7	8
5	48	8	7958
6	5229	9	895536
b = 6		10	27511300
6	622		
7	42464		

B — *Diamond lattice site animals*

s = 1	g_{sb}	s = 5	
0	1	4	91
s = 2		s = 6	
1	2	5	396
s = 3		6	2
2	6	s = 7	
s = 4		6	1782
3	22	7	24

	s = 8	
7		8186
8		207
	s = 9	
8		38199
9		1508
10		6
	s = 10	
9		180544
10		9978
11		102
12		1
	s = 11	
10		862642
11		62112
12		1146
13		16
	s = 12	
11		4161378
12		371001
13		10434
14		198

FIXED SIZE DIRECTED PERCOLATION GROUPINGS

C — *Simple quadratic site problem*

		g_{st}
	s = 1	
2		1
	s = 2	
3		2
	s = 3	
3		1
4		4
	s = 4	
4		5
5		8
	s = 5	
4		2
5		17
6		16
	s = 6	
4		1
5		13
6		50
7		32
	s = 7	
5		10
6		58
7		135
8		64
	s = 8	
5		5
6		57
7		214
8		346
9		128
	s = 9	
5		2
6		45
7		259
8		707
9		854
10		256
	s = 10	
5		1
6		28
7		267
8		1023
9		2163
10		2052
11		512
	s = 11	
6		20
7		218
8		1269
9		3681
10		6264
11		4827
12		1024
	s = 12	
6		10
7		181
8		1278
9		5291
10		12360
11		17383

12	11170
13	2048
$s=13$	
6	5
7	131
8	1219
9	6290
10	20136
11	39329
12	46661
13	25498
14	4096
$s=14$	
6	2
7	90
8	1069
9	6805
10	27455
11	71686
12	119848
13	121873
14	57564
15	8192
$s=15$	
6	1
7	56
8	881
9	6837
10	33337
11	109887
12	242203
13	352343
14	311262
15	128726
16	16384
$s=16$	
7	36
8	681
9	6428
10	37451
11	148186
12	411505
13	784420
14	1005138
15	779932
16	285572
17	32768
$s=17$	
7	20
8	508
9	5741
10	39233
11	183464
12	610686
13	1462141
14	2452215
15	2794187
16	1922948
17	629100
18	65536

C — *Triangular site problem*

$s=1$	g_{st}
3	1
$s=2$	
4	2
5	1
$s=3$	
5	5
6	4
7	1
$s=4$	
5	1
6	12
7	15
8	6
9	1
$s=5$	
6	6
7	31
8	51
9	29
10	8
11	1
$s=6$	
6	2
7	22
8	93

9	162
10	125
11	47
12	10
13	1
s = 7	
7	15
8	77
9	293
10	523
11	485
12	241
13	69
14	12
15	1
s = 8	
7	5
8	65
9	291
10	934
11	1725
12	1800
13	1098
14	407
15	95
16	14
17	1
s = 9	
7	1
8	40
9	265
10	1078
11	3086
12	5739
13	6555
14	4659
15	2114
16	631
17	125
18	16
19	1
s = 10	
8	20
9	199
10	1094
11	3925
12	10452
13	19345

14	23596
15	18901
16	10084
17	3663
18	921
19	159
20	18
21	1
s = 11	
8	6
9	142
10	925
11	4370
12	14317
13	35970
14	66029
15	84536
16	74390
17	45287
18	19350
19	5891
20	1285
21	197
22	20
23	1
s = 12	
8	2
9	75
10	761
11	4144
12	17096
13	52340
14	125301
15	228005
16	302428
17	286950
18	194685
19	95281
20	34057
21	8960
22	1731
23	239
24	22
25	1
s = 13	
9	40
10	522
11	3736

12		17850
13		66212
14		191545
15		441060
16		794995
17		1083076
18		1091816
19		810484
20		444953
21		182225
22		56161
23		13048
24		2267
25		285
26		24
27		1
	s = 14	
9		15
10		348
11		2990
12		17429
13		74526
14		255149
15		701740
16		1565490
17		2796170
18		3886667
19		4116618
20		3294610
21		1994447
22		918464
23		324019
24		88006
25		18349
26		2901
27		335
28		26
29		1

C — *Simple cubic site problem*

	s = 1	g_{st}
3		1
	s = 2	
5		3
	s = 3	
6		3
7		9
	s = 4	
6		1
8		24
9		27
	s = 5	
8		9
9		21
10		126
11		81
	s = 6	
9		15
10		69
11		219
12		567
13		243
	s = 7	
9		3
10		22
11		168
12		571
13		1512
14		2334
15		729
	s = 8	
10		12
11		36
12		394
13		1554
14		4131
15		8598
16		9099
17		2187
	s = 9	
10		3
11		3
12		198
13		798
14		4062
15		12285
16		26619
17		43605
18		34113
19		6561

s = 10	
10	1
12	45
13	426
14	2400
15	10122
16	34907
17	86118
18	155874
19	204408
20	124362
21	19683
s = 11	
12	13
13	153
14	1029
15	6852
16	27480
17	98232
18	265065
19	548817
20	846369
21	905424
22	443484
23	59049
s = 12	
13	48
14	477
15	3156
16	17535
17	82128
18	274809
19	809265
20	1832232
21	3250473
22	4323981
23	3838500
24	1554633
25	177147

C — *Hypercubic 4-dimensional site problem*

s = 1	g_{st}
4	1
s = 2	
7	4
s = 3	
9	6
10	16
s = 4	
10	4
12	66
13	64
s = 5	
10	1
13	52
14	84
15	474
16	256
s = 6	
13	14
15	132
16	514
17	1236
18	2904
19	1024
s = 7	
15	36
16	169
17	286
18	2100
19	5336
20	11922
21	16218
22	4096
s = 8	
16	28
17	82
18	900
19	1770
20	7244
21	25224
22	51254
23	93918
24	85560
25	16384
s = 9	
16	4
18	183
19	686

20	2274
21	13746
22	29603
23	103320
24	259638
25	450758
26	655770
27	433320
28	65536
s = 10	
18	30
19	216

20	224
21	4536
22	13586
23	53986
24	177514
25	439916
26	1196838
27	2413458
28	3640140
29	4214016
30	2130912
31	262144

CYCLOMATIC NUMBER DISTRIBUTIONS (DIRECTED ANIMALS)

D — *Simple quadratic cycle groupings*

s = 3	$g^{(2)}_{sb}$
2	1
s = 4	
3	2
4	1
s = 5	
4	5
5	4
s = 6	
5	14
6	10
7	2
s = 7	
6	38
7	26
8	11
s = 8	
7	100
8	77
9	34
10	5
s = 9	
8	262
9	228
10	102
11	30
12	1
s = 10	
9	690
10	653
11	334

12	105
13	18
s = 11	
10	1818
11	1860
12	1073
13	356
14	98
15	6
s = 12	
11	4790
12	5307
13	3308
14	1277
15	368
16	63
17	2
s = 13	
12	12633
13	15084
14	10087
15	4406
16	1357
17	320
18	36
s = 14	
13	33364
14	42670
15	30638
16	14532
17	5094

18		1291
19		250
20		15
	s = 15	
14		88211
15		120348
16		92290
17		47130
18		18293
19		5126
20		1182
21		164
22		5
	s = 16	
15		233460
16		338642
17		275698
18		151301
19		63146
20		19994
21		4988
22		955
23		98
24		1
	s = 17	
16		618500
17		950692
18		818594
19		479578
20		213949
21		74466
22		20508
23		4476
24		734
25		48

D — *Simple cubic site problem*

	s = 1	g_{sb}
0		1
	s = 2	
1		3
	s = 3	
2		12
	s = 4	
3		49
4		3
	s = 5	
4		204
5		33
	s = 6	
5		870
6		228
7		15
	s = 7	
6		3787
7		1344
8		201
9		7
	s = 8	
7		16722
8		7467
9		1641
10		180
12		1
	s = 9	
8		74643
9		40245
10		11119
11		2037
12		108
13		15
	s = 10	
9		336108
10		212505
11		70752
12		16686
13		2097
14		180
15		18
	s = 11	
10		1524438
11		1105692
12		427305
13		119091
14		22386
15		2740
16		294
17		21

s = 12		15	186237
11	6956214	16	32493
12	5692404	17	3927
13	2498400	18	555
14	794151	20	3

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