ENVELOPE SOLITONS IN A PLASMA STRIP-LINE

A. B. SÁ and J. T. MENDONÇA

Centro de Electrodinâmica
Instituto Superior Técnico, 1096 Lisboa Codex, Portugal

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ABSTRACT — We present some results of a study of the propagation of nonlinear wavepackets in a plasma strip-line system, which can be described as a nonlinear dispersive transmission line.

I – INTRODUCTION

In the last few years much attention has been given to the propagation of solitons in physical systems. In particular, it is now well-known that a plasma can propagate envelope solitons with central frequency nearly equal to the electron plasma frequency. This effect has been studied in uniform [1] and slightly nonuniform [2] semi-infinite plasmas. However, from the experimental point of view it is perhaps more suitable to study a configuration in which the transverse dimension of the plasma is finite. This is the reason why in this paper we discuss the propagation of envelope solitons in a plasma strip-line system. Such a system can be understood as a nonlinear transmission line, where the nonlinearity is associated to the plasma motion.

In Section II we study the plasma strip-line element with which we can construct a transmission line. We will study its properties in the linear approximation, assuming that the plasma electrons are at rest in the absence of an external perturbation. In Section III we take into account the existence of a finite electron temperature. In Section IV we show how a nonlinear transmission line equivalent to a long plasma strip-line can be constructed. In Section V we discuss the equation of propagation along this line. In the linear approximation we obtain the dispersion relation equivalent to that of the line. In the nonlinear regime we show that the equation of propagation for the envelope of a wavetrain can be reduced to the nonlinear Schrodinger equation, if the carrier frequency of the wavetrain is nearly equal to the electron plasma frequency. The soliton solution of the nonlinear Schrodinger equation is then of the Langmuir type. However, the associated electric field in the plasma is perpendicular and not paralell to the direction of propagation, as it is the case in the usual Langmuir solitons. The conclusions are stated in Section VI.

II – COLD PLASMA CONDENSER

We consider a plasma slab of uniform density and thickness a placed between two infinite plane plates P_1 and P_2 (see Figure 1). The distance between plates is l and we apply a potential

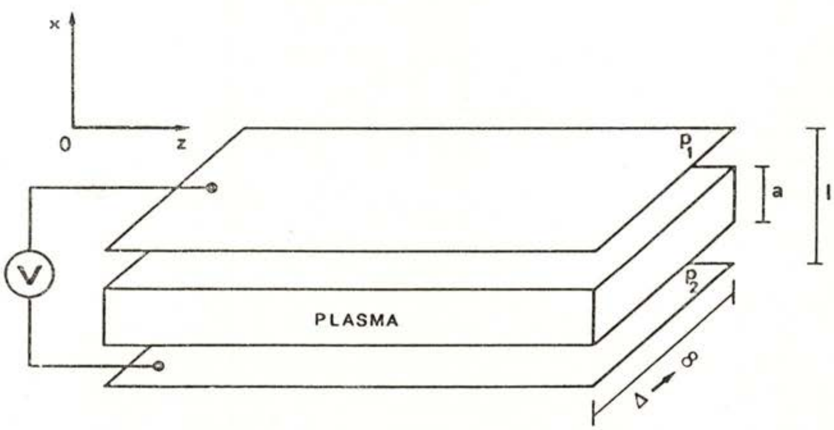

Fig. 1 — Model of the plasma strip-line system.

$V(t) = V_0 \exp(-i\omega t)$ to the plates. The electron motion is described with the aid of hydrodynamic equations which include a kinetic pressure term. The ions are assumed fixed. The electron density n and velocity v are then described by:

$$\begin{aligned} &\partial n / \partial t + \partial (nv) / \partial x = 0 \\ &(\partial / \partial t + \partial v / \partial x)\, v = -(e/m)\, E - \nu_c v - (S_e^2 / n)\, \partial n / \partial x \end{aligned} \qquad (1)$$

where E is the electric field inside the plasma, ν_c is the electron collision frequency and $S_e^2 = 3\,K\,T_e\,/\,m$, where T_e is the electron temperature. The electric field E also obeys the Poisson equation, if electromagnetic corrections are neglected:

$$\partial E / \partial x = (e / \varepsilon_0)\ (n_o - n) \tag{2}$$

where n_0 is the mean electron density. On the other hand, the parameters describing the exterior properties of the plasma strip-line system are the current density J and the potential V between plates, which are determined by:

$$\begin{aligned} J &= \varepsilon_0\ \partial E / \partial t - e\,n\,v = \varepsilon_0 \partial E_e / \partial t \\ V &= E_e\ (l - a) + \int_{-a/2}^{+a/2} E\ dx \end{aligned} \tag{3}$$

where E_e is the electric field outside the plasma, in the vacuum region laying between the plasma and each plate.

Let us now take the cold plasma approximation ($T_e = S_e = 0$). In that case we assume homogeneity along x and, after time Fourier analysis of equations (1) - (3) we find, for each component ω of the Fourier spectrum,

$$\begin{aligned} J &= -i\,\omega\,\varepsilon_0\ E_e = -i\,\omega\,\varepsilon_0\,\varepsilon(\omega)\ E \\ V &= [\,(l - a)\ \varepsilon(\omega) + a\,]\ E \end{aligned} \tag{4}$$

where $\varepsilon(\omega)$ is the cold plasma dielectric constant:

$$\varepsilon(\omega) = 1 - (\omega_p / \omega)^2\ (1 + i\,\nu_c / \omega)^{-1} \tag{5}$$

where $\omega_p = (e^2\,n_0 / \varepsilon_0\,m)^{1/2}$ is the electron plasma frequency. From equation (4) we get then the impedance of the plasma strip-line, per unit area,

$$Z = V / J = [\,(l - a) + a\,\varepsilon(\omega)^{-1}\,] / (-i\,\omega\,\varepsilon_0) \tag{6}$$

It is easy to see that this impedance is composed by a series of two capacities C_v and C_ω, such that:

$$C_v = \varepsilon_0 / (l - a)\quad,\quad C_\omega = {}_0\,\varepsilon(\omega) / a \tag{7}$$

Replacing (5) in the equation for C_ω we obtain circuit elements which do not depend on the frequency ω. In fact, the impedance Z_ω associated to C_ω can be writen as:

$$Z_\omega = i / (\omega C_\omega) = (R_p - i \omega L_p) / (1 - \omega^2 L_p C_p - i \omega R_p C_p) \tag{8}$$

where the resistance R_p, the inductance L_p and the capacity C_p are defined by:

$$R_p = a \nu_c / (\varepsilon_o \omega_p^2) \quad ; \quad L_p = a / (\varepsilon_o \omega_p^2) \quad , \quad C_p = \varepsilon_o / a \tag{9}$$

Thus Z_ω is a parallel R L C circuit and the plasma condenser is equivalent to such circuit in series with C_v.

A resonance $(Z = 0)$ occurs for

$$\omega = \omega_p \sqrt{1 - (a / l)} \tag{10}$$

as can be seen from equation (6) if the damping terms are neglected; and an anti-resonance $(Z \rightarrow \infty)$ for $\varepsilon \rightarrow 0$, which, in the collisionless limit, leads to $\omega = \omega_p$.

III – HOT PLASMA CONDENSER

We consider now the situation where $T_e \neq 0$. In this case we can no longer neglect the spatial perturbation in the x direction. Making a Fourier transform in time, we get from equation (1) the following expressions for the electron density and velocity perturbations:

$$\begin{aligned} \tilde{n} &= n_o / (i \omega) \cdot \partial v / \partial x \\ v &= (1 / i \omega) \, (1 + i \nu_c / \omega)^{-1} \, (e E / m + S_e^2 / n_o \cdot \partial \tilde{n} / \partial x) \end{aligned} \tag{11}$$

where $\tilde{n} = n - n_o$.

Using equation (2) and after eliminating v and n, we obtain the following equation for the electric field:

$$[\partial^2 / \partial x^2 + k^2] \, \partial E / \partial x = 0 \tag{12}$$

where

$$k^2 = [\omega^2 (1 + i \nu_c / \omega) - \omega_p^2] / S_e^2 \tag{13}$$

The general solution of this equation is of the form:

$$E = E_o + A \cos kx + B \sin kx \qquad (14)$$

Assuming now that the velocity at the plasma boundary $(x = \pm a/2)$ is equal to zero, we get from the Fourier transform of equation (3a) the following result:

$$B = 0 \quad , \quad E_o + A \cos(ka/2) = E_e \qquad (15)$$

This means that the field (14) reduces to a purely time varying field E_0 plus a space dependent field of cosinus form. Returning to equation (11) we get:

$$\begin{aligned} \tilde{n} &= (\varepsilon_o k/e)\, A \sin kx \\ v &= (e/m)\,(1/i\omega)\,(1 + i\nu_c/\omega)^{-1} \,. \\ &\quad \cdot [E_o + (1 + k^2 S_e^2/\omega_p^2)\, A \cos kx] \end{aligned} \qquad (16)$$

Using once more the assumption that the velocity is zero at $x = \pm a/2$ we get from equations (15b) and (16b):

$$\begin{aligned} A &= -(\omega_p/k S_e)^2\, E_e \sec(ka/2) \\ E_o &= E_e\,(1 + \omega_p^2/k^2 S_e^2) \end{aligned} \qquad (17)$$

We can now express the external potential drift V as a function of E_e. Using (17) in the Fourier transform of equation (3) we get:

$$V = E_e\,[\,l + a\,\omega_p^2/(k^2 S_e^2) - 2\,\omega_p^2/(k^3 S_e^2)\tan(ka/2)\,] \qquad (18)$$

The plasma condenser impedance can be easily obtained, if we use $J = -i\,\omega\,\varepsilon_o E_e$:

$$Z = -i/(\omega\,\varepsilon_o)\ .\ \omega_p^2/(k^3 S_e^2)\,[-l\,k^3 S_e^2/\omega_p^2 - ka + 2\tan(ka/2)] \qquad (19)$$

This equation as well as equation (6), are well known in the literature [3], but here we have used a more straightforward calculation. The anti-resonances of the system $(Z \to \infty)$ are now given by the condition $\cos(ka/2) = 0$, which leads to:

$$\omega_N = \omega_p\,[1 + (2N+1)^2\,\pi^2\,\lambda_D^2/a^2]^{1/2} \qquad (20)$$

where $N = 0, 1, 2, \ldots$ and $\lambda_D^2 = S_e^2 / \omega_p^2$. The resonances ($Z \rightarrow 0$) are obtained by solving the equation (cf. [3]):

$$\tan (ka / 2) = (ka / 2) + 1/2 \; l \; k^3 \; \lambda_D^2 \qquad (21)$$

When $\omega < \omega_p$ the wavenumber becomes imaginary (cf. eq. (13)); putting $z = -i\,ka / 2$ we can rewrite eq. (21) in the form:

$$\tanh z - z + z^3 \; 4 \, l \; \lambda_D^2 / a^3 = 0 \qquad (22)$$

When $| ka | << 1$ an expansion of eq. (21) or (22) leads to:

$$\omega = \omega_p \; [1 - 10 \; \lambda_D^2 / a^2 + 120 \; l \, \lambda_D^4 / a^5]^{1/2} \qquad (23)$$

In the general case (21) has solutions k_N ($N = 1, 2, \ldots$) such that

$$k_N \; a / 2 = x_N + \delta_N \qquad (24)$$

where x_N is the Nth non-zero root of $\tan x = x$ and $\delta_N \rightarrow 0$ when $\lambda_D << a$ ($x_N + \delta_N \rightarrow (2N + 1) \pi / 2$ when $\lambda_D >> a$). An additional solution exists, given by (21) or (22) as $4 \, l \, \lambda_D^2 / a^3$ is larger or smaller than 1/3, respectively; when $\lambda_D << a$ then $z^2 \rightarrow a^3 / (4 \, l \, \lambda_D^2)$.

Comparing the results with those obtained in the previous section we see that the influence of the temperature is to replace the anti-resonance $\omega = \omega_p$ by an infinite number of anti-resonances $\omega = \omega_N$ (eq. (20)), which when $\lambda_D << a$ (corresponding to the usual situation in laboratory experiments) lie close to ω_p. On the other hand the resonance $\omega_p [1 - (a / l)]^{1/2}$ is also replaced by an infinite number of resonances. Since, from (13),

$$\omega = \omega_p [1 + (2\lambda_D / a)^2 \; (k \; a / 2)^2]^{1/2} \qquad (25)$$

it is easy to see that, for $\lambda_D << a$, such resonances (eq. (24)) lie close to ω_p, but the "additional solution" lies close to $\omega_p [1 - (a / l)]^{1/2}$.

If the nonlinear terms of equations (1) - (3) are now taken into account we get an expression for the impedance Z which is formally analogous to equation (19) but where ω_p is replaced by an effective plasma frequency which depends on the square amplitude of the potential:

$$\omega_{eff}^2 = \omega_p^2 \; (1 - \alpha \, | V |^2) \qquad (26)$$

The perturbative nonlinear analysis is quite lengthy and will not be presented here. We just quote the approximate value for the parameter α [4]:

$$\alpha = \alpha' \{ 1 - 1/3 \; (\omega_p / kS_e)^4 \; [1 + (kS_e / \omega_p)^2]^2$$
$$[1/2 + 2 \; (\omega_p / \omega)^2 \; (1 + (kS_e / \omega_p)^2)] \} \qquad (27)$$

with $\alpha' = \varepsilon_0 / (2 n_o T_e a^2)$

This nonlinear parameters can be justified in a rather simple way. If we take the equation of motion in the x direction and average over a time scale of the order of $1 / \omega_p$, we get for the mean velocity the following equation

$$\partial < v > / \partial t = -(e / m) \cdot \partial / \partial x \; (\varepsilon_0 |E|^2 / e n_o) \qquad (28)$$

This equation shows that there exists an effective potential V_{eff} acting on the electrons and making them move (in a time scale much larger than $1 / \omega_p$):

$$V_{eff} = \varepsilon_0 |E|^2 / (e n_o) \qquad (29)$$

Assuming that the electrons reach thermodynamic equilibrium in this potential [1], we get for the mean density:

$$< n > = n_o \exp(-eV_{eff} / K T_e) \qquad (30)$$

Assuming now that $eV_{eff} << KT_e$, the mean electron density will be given by:

$$< n > = n_o \; (1 - eV_{eff} / K T_e) \qquad (31)$$

Using (29) and considering that $|E|^2$ is proportional to $|\tilde{V}|^2$ we can see that this nonlinear correction to the mean electron density is of the form $-\alpha |\tilde{V}|^2$, as stated above.

IV – EQUIVALENT TRANSMISSION LINE

We are now able to define the nonlinear transmission line, which is an electric analog of the strip-line plasma system. This line can be viewed as an infinite series of condenser elements, each of which is described by the equations deduced in the previous section. In order to have a complete description of the line we must add an inductance L_l, and a resistance R_l has to be retained when we consider the propagation along z. We can easily obtain the equivalent circuit of fig. 2, where the cold plasma limit was considered. The plasma inductance $L_p(V^2)$ is given by equation (9) where ω_p^2 was replaced by ω_{eff}^2. In the expression of R_p we neglect the nonlinear corrections to the plasma frequency, because R_p is already a small quantity. In this work we will be interested only on the complete case of a transmission line without losses ($R_l = R_p = 0$). We will also take $l = a$ in the configuration of figure 1, which means $C_v = \infty$. It can be shown [4] that in the most general case the nonlinear solutions are quite similar to those obtained here. In the assumed

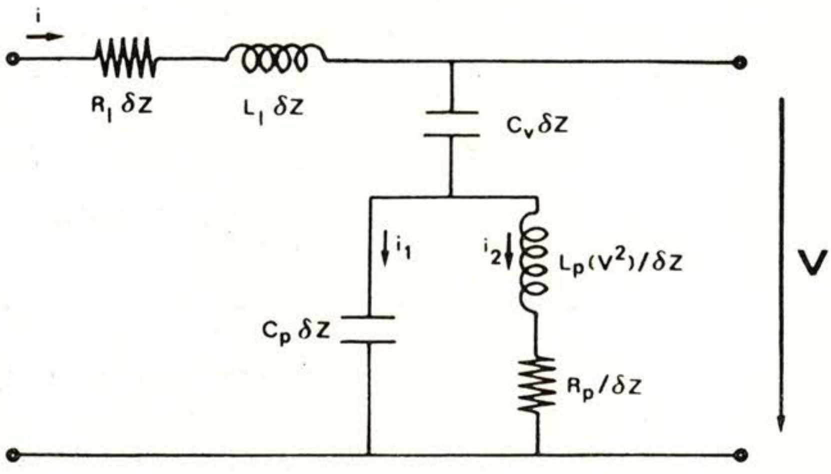

Fig. 2 — Equivalent electric circuit to the transmission line.

approximation we can see from figure 2 that the current flowing in the circuit elements is given by:

$$\partial i / \partial t = (1 / L_l)\, \partial V / \partial z \qquad (32)$$

On the other hand we also have:

$$\delta i = (\partial i / \partial z)\ \delta z = i_1 + i_2 \qquad (33)$$

where the currents i_1 and i_2 are given by:

$$i_1 = C_p \, \delta z \cdot \partial V / \partial t \quad , \quad i_2 = (\delta z / L_p) \int V dt \qquad (34)$$

From these equations we can easily get the equation of the potential perturbation along the line:

$$\partial^2 V / \partial t^2 - (1 / C_p L_l) \, \partial^2 V / \partial z^2 + V / (C_p L_p) = 0 \qquad (35)$$

V – ENVELOPE SOLITONS

Let us consider now a potential of the form:

$$V (z , t) = \tilde{V} (z , t) \; e^{-i \omega t} \qquad (36)$$

where $\tilde{V} (z , t)$ is a slowly varying amplitude, in the sense that:

$$| \partial \tilde{V} / \partial t | \; << \; \omega \; | \tilde{V} | \qquad (37)$$

Replacing (36) in (35) and taking (37) into account we get an equation of propagation for the envelope in the form:

$$- 2 i \omega \, \partial \tilde{V} / \partial t - \omega^2 \tilde{V} - (1 / C_p L_l) \, \partial^2 \tilde{V} / \partial z^2 + \tilde{V} / (C_p L_p) = 0 \qquad (38)$$

If we use now the nonlinear expression for the plasma inductance L_p we get:

$$1 / (C_p L_p) = \omega_p^2 \, (1 - \alpha \, | \tilde{V} |^2) \qquad (39)$$

Assuming that the wave frequency ω is nearly equal to the electron plasma frequency, equation (38) reduces to:

$$2 i \omega \; \partial \tilde{V} / \partial t + \omega^2 \alpha \, | \tilde{V} |^2 \; \tilde{V} + (1 / C_p L_l) \, \partial^2 \tilde{V} / \partial z^2 = 0 \qquad (40)$$

Using now space and time adimensional variables:

$$\tau = (\omega / 2) \, t \quad , \quad \mu = z \, \omega \, (C_p L_l)^{1/2} = z \, (L_l / L_p^*)^{1/2} \qquad (41)$$

where L_p^* is the linearised plasma inductance we finally obtain:

$$i \, \partial \tilde{V} / \partial \tau + \partial^2 \tilde{V} / \partial \mu^2 + \alpha | \tilde{V} |^2 \tilde{V} = 0 \tag{42}$$

This is the well known nonlinear Schrodinger equation which, for $\tilde{V}$ tending to zero at infinity ($\tilde{V} \rightarrow 0$ for $z \rightarrow \infty$) has the following soliton solution [5]:

$$\tilde{V} = A (2 / \alpha)^{1/2} \exp \{ i [(B / 2) \mu - (B^2 / 4 - A^2) \tau] \} \operatorname{sech} [A (\mu - B \tau)] \tag{43}$$

where A and B are two constants of integration. The first constant A defines the amplitude of the soliton perturbation and the second one B defines the velocity at which this perturbation moves along the line. We can then specify B, because it has to be equal to the usual group velocity v_g in the coordinates μ and τ. In order to determine v_g we return to equation of propagation (35). After linearization and using $V = V_0 \exp i (kz - \omega t)$ we get the linear dispersion relation of the line which describes the evolution of each Fourier component of the soliton spectrum.

$$\omega^2 = \omega_p^2 + k^2 / (C_p L_l) \tag{44}$$

The phase and group velocities along the line (in the coordinates z and t) are given by:

$$\begin{gathered} v_\varphi = \omega / k = \omega_p [1 / k^2 + 1 / (C_p L_l \omega_p^2)]^{1/2} \\ v_g = 1 / (C_p L_l v_\varphi) \end{gathered} \tag{45}$$

We can then state the explicit form of the constant B:

$$B = 2 \, [1 + \omega_p^2 C_p L_l / k^2]^{-1/2} \tag{46}$$

This equation completely specifies the soliton solution of (43).

VI – CONCLUSIONS

We have shown in this work that a long plasma strip-line system can be described by an equivalent transmission line. This line has nonlinear properties, which are associated with the nonlinear motion of the plasma particles induced by the potential applied to the transmission line. We have determined the linear dispersion relation of the line and as our main result, we have shown that such a line can propagate envelope solitons, which are similar to the well known Langmuir solitons. However the nonlinear equation of propagation along the line differs slightly from the Zakharov equation which describes the Langmuir solitons in an unbounded plasma. The main difference is that our solitons are solitons in the strict sense, as defined by Scott et al. [6], and the usually called one dimensional Langmuir solitons are solitary waves which are not solitons in this sense [7]. In the case of a line with a finite resistance R_p we can also get soliton solutions propagating along the line with a slight damping [8].

This work remains valid only in the limit of low electronic temperatures. In the case of finite temperatures we have to compare the spectral width of the soliton solution with the distance between two neighbouring resonances in order to conclude about the validity of the previous results. However it is quite obvious that in the general case the nonlinear equation of propagation cannot be written in the simple form used here. The general features of soliton propagation using a consistent theory for a finite temperature plasma will be discussed elsewhere.

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