A SIMPLE MODEL FOR THE PION FIELD AROUND A STATIC SOURCE

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ABSTRACT — We present a simple model for a system of static bare nucleon and bare delta, coupled linearly to a non-self-interacting pion cloud. The model space for the boson cloud consists of states with arbitrary number of $l = 1$ pions, having all the same radial wave function. Both axially symmetric and hedgehog coherent states of pions are studied, in order to compare their behaviour as a function of the coupling strength and to look at the relevance of the variation-after-projection method. The model may be used as a test of different approximations commonly applied in realistic calculations for meson clouds.

1 — INTRODUCTION

Several chiral invariant models have been proposed to describe the pion cloud around the nucleon: the chiral bag [1], the little bag [2], the cloudy bag [3] and the chiral soliton [4]. A fully quantum mechanical treatment of these models is possible only in the framework of perturbation theory. On the other hand, in the non-perturbative regime the solutions are obtained in the mean-field-approximation, assuming the so-called hedgehog form for the fields. These solutions are not eigenfunctions of the angular

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momentum and the isospin operators and, therefore, they cannot directly describe the physical states. However, as it is well known from nuclear physics, the states with good spin and isospin quantum numbers may be obtained from them by means of the Peierls-Yoccoz projection technique [5].

In this paper we refer to a system of static bare nucleon and bare delta, coupled linearly to a non-self-interacting pion cloud, e. g. the cloudy-bag model [3]. We present a very simple model which is suitable for testing the validity of different approximation schemes to the meson cloud. In a given realistic model one should estimate the number of pions *n* and then, one can get some insight from the simple model, how different approximations behave in that range of *n*.

The contents of this paper are as follows. In section 2 the model is presented. The regimes of weak and strong coupling strengths are considered in section 3. Angular momentum and isospin projections from axially symmetric coherent states and hedgehog coherent state of pions are performed in sections 4 and 5. Section 6 contains the discussion of the different approximate solutions and the conclusions. The technical details of the spin-isospin projection are presented in the appendices.

2 — THE MODEL

For a system of p-wave pions interacting with static bare particles, the Hamiltonian can be written in the form [6]

$$H' = \sum_a \varepsilon_a c_a^+ c_a + \sum_{tmk} \{ \omega(k)\ a_{tm}^+(k)\ a_{tm}(k) - \tilde{G}\, \rho(k)\ B_{tm} [a_{tm}(k) + (-1)^{t+m} a_{-t-m}(k)] \} \tag{2.1}$$

where c_a^+ is the creation operator for the bare particle and ε_a its energy; the operator $a_{tm}^+(k)$ creates a pion with momentum (magnitude) k, angular momentum one and (spherical) angular momentum and isospin components $m = 0, \pm 1$ and $t = 0, \pm 1$, respectively. We consider a model with bare nucleon and bare delta. $\varepsilon_N = \varepsilon_\Delta = 0$ will be assumed. In (2.1) the operator B_{tm} is given by [6]

$$B_{tm} = \tau_t^{NN} \sigma_m^{NN} + \sqrt{72/25} \left(\tau_t^{N\Delta} \sigma_m^{N\Delta} + \tau_t^{\Delta N} \sigma_m^{\Delta N} \right) + (4/5) \tau_t^{\Delta\Delta} \sigma_m^{\Delta\Delta} \tag{2.2}$$

where τ_t^{NN} and σ_m^{NN} are the Pauli matrices acting on the isospin and the spin of the bare nucleon in the source; the operators $\tau_t^{\Delta\Delta}$ and $\sigma_m^{\Delta\Delta}$ do the same on bare delta; $\tau_t^{N\Delta} \sigma_m^{N\Delta}$ converts a bare delta in bare nucleon and $\tau_t^{N\Delta} \sigma_m^{N\Delta}$ vice-versa.

We shall not specify the spherically symmetric source density $\rho(k)$ in (2.1). However, for a given $\rho(k)$, the one pion radial wave function $F(k)$ can be determined by a self consistent mean-field calculation of the intrinsic state or, in a more reliable calculation, it can be determined variationally for a projected coherent state.

The creation operator $a_{tm}^+(k)$ can always be written in the form

$$a_{tm}^+(k) = F^*(k)\, a_{tm}^+ + \sum_n F_n^*(k)\, b_{tm}^+(n) \tag{2.3}$$

where a_{tm}^+ and $b_{tm}^+(n)$ form a complete orthonormal set. Here, a_{tm}^+ creates a pion with angular momentum and isospin components m and t, respectively, and radial wave function $F(k)$; $b_{tm}^+(n)$ creates a pion with the same angular momentum and isospin quantum numbers but with radial wave function $F_n(k)$. These states are not occupied in the model space and therefore all $b_{tm}^+(n)$ can simply be ignored in the Hamiltonian.

In this paper we shall study the properties of approximate solutions for the Hamiltonian

$$H = \sum_{tm} \left\{ a_{tm}^+ a_{tm} - G B_{tm} \left[a_{tm} + (-1)^{t+m} a_{-t-m}^+ \right] \right\} \tag{2.4}$$

with

$$G = \tilde{G} \sum_k \rho(k)\, F(k) \tag{2.5}$$

in energy units in which $\Sigma_k\, \omega(k)\, F^*(k)\, F(k) = 1$.

The Hamiltonian H is schematic in the sense that it contains no radial degree of freedom. It may be viewed as an effective Hamiltonian which is replacing (2.1).

3 – WEAK AND STRONG COUPLINGS

In the limit of small G, the second term in (2.4),

$$H_{coup} = -G \sum_{tm} B_{tm} \, (a_{tm} + (-1)^{t+m} + a^{+}_{-t-m}) \qquad (3.1)$$

may be considered as a perturbation to the first one. In this regime, the physical baryon states are superpositions of bare baryon, $| \phi_0 >$, and bare-baryon-plus-one-pion states, $| \phi_i >$. Up to second order in the coupling constant, the energy is

$$E^{pert} = E_0 + \varepsilon_1 + \varepsilon_2 \qquad (3.2)$$

where E_0 is the bare baryon energy,

$$\varepsilon_1 = < \phi_0 | \, H_{coup} \, | \phi_0 > \qquad (3.3)$$

and

$$\varepsilon_2 = \sum_i \, | < \phi_0 | \, H_{coup} \, | \phi_i > |^2 / (E_0 - E_i) \qquad (3.4)$$

In our simple model no chromomagnetic interaction is considered and for simplicity we have taken as zero both bare nucleon and the bare delta energies. Moreover, for the Hamiltonian (2.4), $E_i = 1$. From (3.1) and (3.3) it follows that the first order contribution for the energy, ε_1, vanishes. The evaluation of (3.4) yields for the energies of the physical nucleon and delta states the following results:

$$E_N = -20.52 \; G^2 \qquad (3.5)$$

and

$$E_\Delta = -11.88 \; G^2 \qquad (3.6)$$

where all these factors are exact fractions (multiples of 1/25).

In the other limit, i. e. in a regime of very strong coupling strength, the cloud around the static source contains a large number of non-self-interacting pions and therefore the mean-field-approximation (MFA) is totally adequate. This consists in describing the pions by quantum mechanical coherent states [7]. In the MFA, the minimal energy for the Hamiltonian (2.4) is obtained for the hedgehog baryon configuration [8] and, as we shall see in section 5, is given by

$$E = -9.72 \; G^2. \qquad (3.7)$$

In a regime of very large G, this is the leading term for the energy of the nucleon and its isobars. The quantum fluctuations are very small and all these states have the same energy.

As a starting point to the description of the pion cloud around the static source we shall consider the mean-field or coherent state approximation. The total trial wave function of the baryon reads as

$$| \psi > = \mathcal{N}(\xi) \, \exp\big(\sum_{tm} \xi_{tm} \, a^{+}_{tm}\big) \, | B > \qquad (3.8)$$

where $\mathcal{N}(\xi)$ is a normalization factor, $| B >$ is the bare baryon state and ξ_{tm} are amplitudes to be determined variationally. The state (3.8) has the following important property:

$$a_{tm} \, | \psi > = \xi_{tm} \, | \psi > . \qquad (3.9)$$

In the MFA, the energy is the expectation value of H in the state (3.8). Using (3.9) this 'intrinsic energy' is readily evaluated yielding

$$E^{(intr.)} = E^{(intr.)}_{kin} + E^{(intr.)}_{coup} , \qquad (3.10)$$

where

$$E^{(intr.)}_{kin} = \sum_{tm} \overset{*}{\xi}_{tm} \, \xi_{tm} \qquad (3.11)$$

and

$$E^{(intr.)}_{coup} = - G \sum_{tm} \big(\xi_{tm} + (-1)^{t+m} \, \overset{*}{\xi}_{-t-m} \big) \, v_{tm} , \qquad (3.12)$$

are respectively the intrinsic kinetic and interaction energies. Here, v_{tm} is the matrix element:

$$v_{tm} = < B \, | \, B_{tm} \, | \, B > , \qquad (3.13)$$

where B_{tm} is the operator (2.2). We notice that for the Hamiltonian (2.4) the kinetic energy is equal to the number of pions in the cloud.

The amplitudes ξ_{tm} are obtained performing a Ritz's variation, $\partial E^{(intr.)} / \partial \xi^*_{tm} = 0$, yielding

$$\xi_{tm} = (-1)^{t+m} G \; v_{-t-m} \; . \tag{3.14}$$

The solutions obtained in the MFA clearly violate the rotational symmetry of the Hamiltonian (2.4) both in coordinate and charge spaces. The trial coherent state (3.8) does not have the spin and isospin quantum numbers of the nucleon or its isobars. These physical states may be obtained by means of a Peierls-Yoccoz projection of the trial wave function onto states of good spin and isospin [5, 6, 9-11].

4 – PROJECTION FROM AXIALLY SYMMETRIC STATES

In this section we consider the projection from axially symmetric coherent states of pions. We assume the bare baryon state to be a mixture of a bare nucleon and bare delta, each having spin component 1/2 and charge +1:

$$| B(\delta) >_A = \cos\delta \; | N^{+}_{1/2} > + \sin\delta \; | \Delta^{+}_{1/2} > \tag{4.1}$$

Here, the mixing angle δ is another variational parameter. Now the amplitude (3.14) reads as

$$\xi_{tm} = \xi \; \delta_{to} \; \delta_{mo} \tag{4.2}$$

where

$$\xi = G \; [\cos^2\delta + (8\sqrt{2}/5) \; \sin\delta \; \cos\delta + (1/5) \; \sin^2\delta] \tag{4.3}$$

From (3.10-12) and (4.2-3) the kinetic, coupling and total intrinsic energies are given by:

$$E^{(intr.)}_{kin} = S = \xi^2 = G^2 \; f(\delta) \; , \tag{4.4a}$$

$$E^{(intr.)}_{coup} = -2\,G\,\xi\sqrt{f(\delta)} = -2\,G^2 \; f(\delta) \; , \tag{4.4b}$$

$$E^{(intr.)} = -G^2 \; f(\delta) \; , \tag{4.4c}$$

where

$$f(\delta) = [\cos^2\delta + (8\sqrt{2}/5)\sin\delta\cos\delta + (1/5)\sin^2\delta]^2 \qquad (4.5)$$

Here and in the sequel S denotes the average (intrinsic) number of pions in the cloud. This quantity is plotted in Figure 1, in dependence of G. The function (4.5) is plotted in Fig. 2 of ref [10].

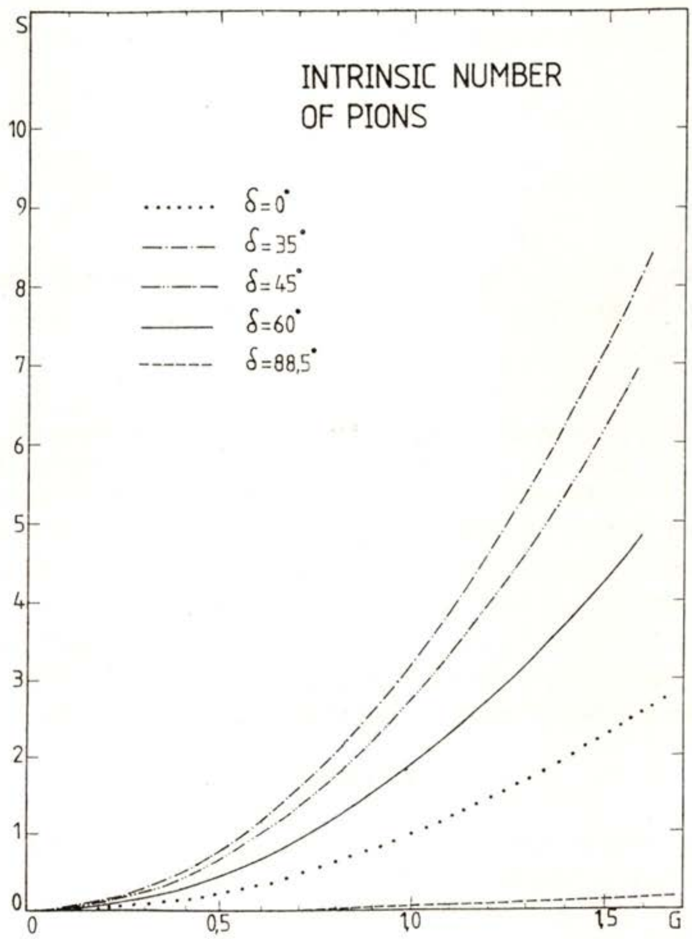

Fig. 1 — The average number of pions in the cloud of the non-projected axially symmetric state. The values are given in dependence of the coupling constant for different mixing angles.

It has an absolute maximum (a minimum for the energy) for $\delta = \text{arctg}\,(1/\sqrt{2}) \simeq 35^{\circ}$, and the corresponding total energy is

$$E^{(\text{intr.})} = -3.24\ G^2 . \tag{4.6}$$

Other values of $E^{(\text{intr.})} / G^2$, for several mixing angles, may be read directly from figures 3 and 5.

Let us use the following ansatz for the total trial wave function of the baryon:

$$|\psi> = \cos\delta\ |\psi_N> + \sin\delta\ |\psi_\Delta> ; \tag{4.7}$$

where

$$|\psi_N> = \exp(\xi\, a_{00}^{+})\ |N_{1/2}^{+}> \tag{4.8a}$$

and

$$|\psi_\Delta> = \exp(\xi\, a_{00}^{+})\ |\Delta_{1/2}^{+}> \tag{4.8b}$$

The state with the quantum numbers of a nucleon is extracted from (4.7) according to

$$|1/2,\ 1/2> = P_{1/2\ 1/2}\,|\psi> \tag{4.9}$$

where $P_{1/2\ 1/2}$ is used as an abbreviation for the operator $P^T_{1/2\ 1/2\ 1/2}$ $P^J_{1/2\ 1/2\ 1/2}$; here, $P_{t\alpha\alpha'}$ projects out of a state with good third component α' of isospin (angular momentum), the state with good quantum number of isospin (angular momentum) t and rotates the third component into α. The norm (square) of the state (4.9) is $F = \cos^2\delta\ F_N + \sin^2\delta\ F_\Delta$, where $F_N = <\psi_N\,|\,P_{1/2\ 1/2}\,|\,\psi_N>$ and $F_\Delta = <\psi_\Delta\,|\,P_{1/2\ 1/2}\,|\,\psi_\Delta>$. The details of the calculation of these quantities are shown in Appendix A.

The number of pions in the cloud – see (A. 15) – is

$$n = S\,F^{-1}(S)\,F'(S) . \tag{4.10}$$

F' denotes the derivative with respect to S. Using the property (3.9), the interaction energy is readily evaluated and the total projected energy for the nucleon reads as

$$E_{1/2\ 1/2} = SF^{-1}\ F' - 2GF^{-1}\ \xi\ [\cos^2\delta\ F_N + (4\sqrt{2}/5)\ \sin\delta\ \cos\delta\ (F_N + F_\Delta) + (1/5)\ \sin^2\delta\,] \tag{4.11}$$

The energy of the physical nucleon is a function of the two variational parameters. Now, the appropriate procedure is to search for a minimum of $E_{1/2\,1/2}$ in the plane (ξ, δ). This is nothing but the variation-after-projection (VAP) method. Another procedure, the variation-before-projection (VBP) method, would be to insert in (4.11) the self-consistent values for the variational parameters obtained in the MFA. We shall study the behaviour of the solutions obtained in both methods.

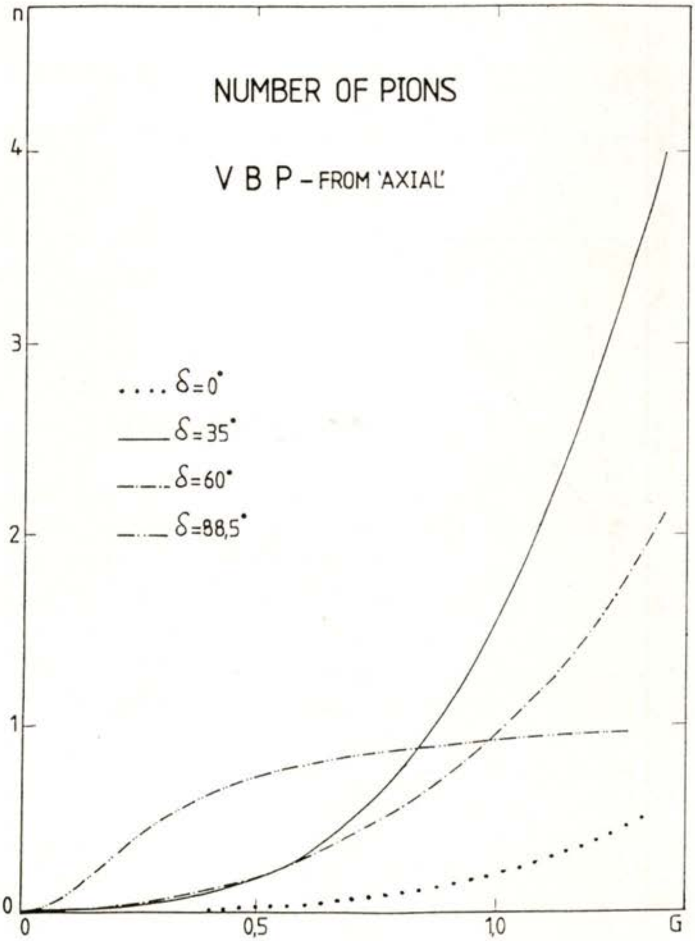

Fig. 2 — The number of pions in the pion cloud of the nucleon, for several mixing angles. The projected states have been obtained from axially symmetric coherent states of pions.

4.1 — *Variation-before-projection*

We have taken the self-consistent ξ, given by (4.3), and have considered the solutions for different values of δ.

The number of pions and the total energy of the projected state have been evaluated numerically from (4.10) and (4.11) using the power series expansions derived in Appendix A, for the norms of the projected coherent states. Figure 2 and 3 show the projected

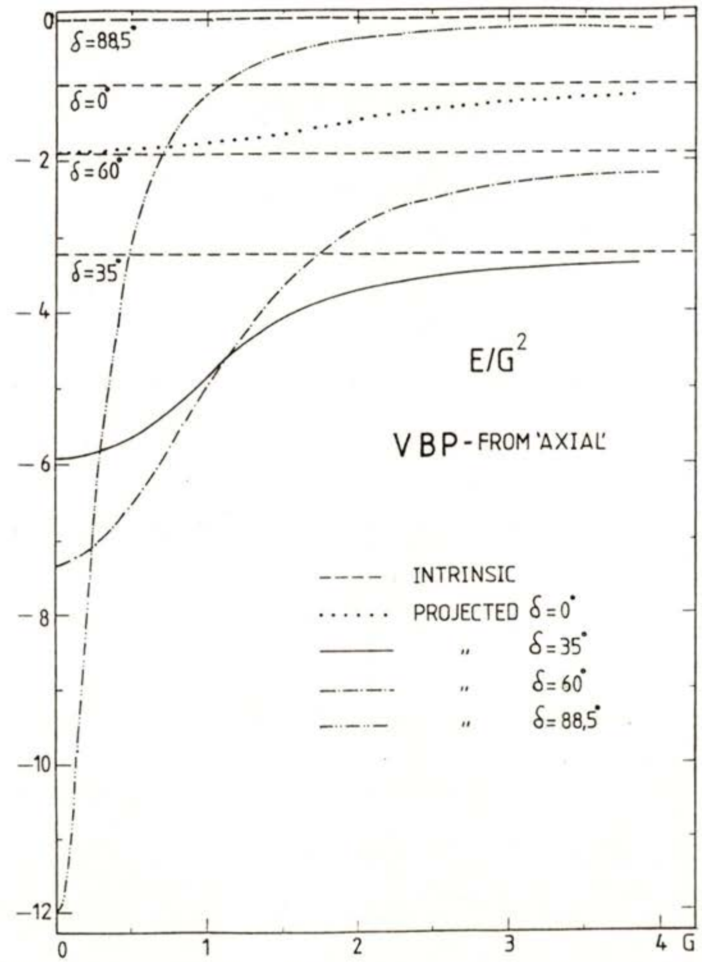

Fig. 3 — For the nucleon, E/G^2 is plotted versus coupling constant for various values of δ. The projected states have been obtained from axially symmetric coherent states of pions.

number of pions and $E_{1/2\,1/2} / G^2$, in dependence of G, for several mixing angles. A comparison between figures 1 and 2 shows that the number of pions gets reduced with the projection. The only exception is observed for angles close to 90°: the source is mainly bare delta and at least one pion is necessary to construct a state with the quantum numbers of a nucleon, whereas for the physical delta the main contribution comes from the zero pion state. Regarding the total energy, for very small values of the coupling constant, the best angle is also around 90°: the minimal energy in the limit $G \rightarrow 0$ is

$$\lim_{G \to 0} E_{1/2\,1/2} / G^2 = -12.02 \,, \qquad (4.12)$$

with $\delta = 88.5°$. This value is very far from (2.5), which is exact in this limit.

4.2 — *Variation-after-projection*

This procedure is more reliable, since it assumes that the eigenstate of the Hamiltonian is approximated by the trial wave function (4.9) which has already the quantum numbers of the nucleon.

For the projected energy (4.11), the variation with respect to ξ, $\partial E_{1/2\,1/2} / \partial \xi = 0$, yields

$$G = \xi \, T(S) \qquad (4.13)$$

where

$$T(S) = (F F' - S F'^2 + S F F'') \cdot \{ (F - 2SF') \cdot$$
$$[\cos^2 \delta \, F_N + (4\sqrt{2}/5) \sin \delta \cos \delta \, (F_N + F_\Delta) + (1/5) \sin^2 \delta \, F_\Delta] +$$
$$2SF [\cos^2 \delta \, F'_N + (4\sqrt{2}/5) \sin \delta \cos \delta \, (F'_N + F'_\Delta)$$
$$+ (1/5) \sin^2 \delta \, F'_\Delta] \}^{-1} \,. \qquad (4.14)$$

The numerical evaluation of the projected energy has been carried out in the following way: First ξ $(= \sqrt{S})$ was fixed; then, using (4.13-14) the coupling constant was determined; finally the number of pions and the total projected energy were evaluated making use of the expressions (4.10-11).

Fig. 4 shows the number of pions in the cloud as a function of coupling strength, for different values of the mixing angle. In Figure 5 the quantity $E_{1/2\ 1/2} / G^2$ is plotted against G. This figure should be compared with Figure 3, where the curves have

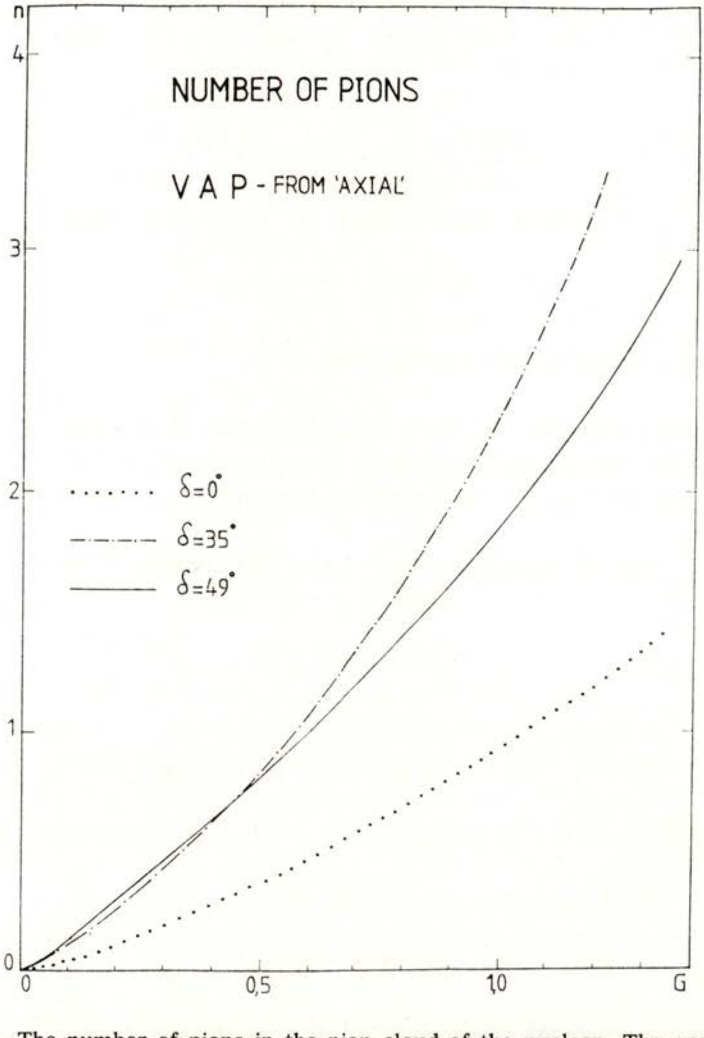

Fig. 4 — The number of pions in the pion cloud of the nucleon. The projected states have been obtained from axially symmetric coherent states of pions in a variation-after-projection calculation with respect to the pion amplitudes.

been obtained in a VBP calculation, in order to see the importance of the VAP procedure, for small values of the coupling constant.

The determination of the optimal parameters ξ and δ and, afterwards, the evaluation of $E_{1/2\ 1/2}/G^2$, may be done analytically for $G \rightarrow 0$. The minimization of $E_{1/2\ 1/2}$ with respect to ξ and δ yields $\delta = \mathrm{arctg}\ (4\sqrt{2}/5)$, $\xi = 9$ G and the energy is

$$E_{1/2\ 1/2} = -20.52\ G^2. \qquad (4.15)$$

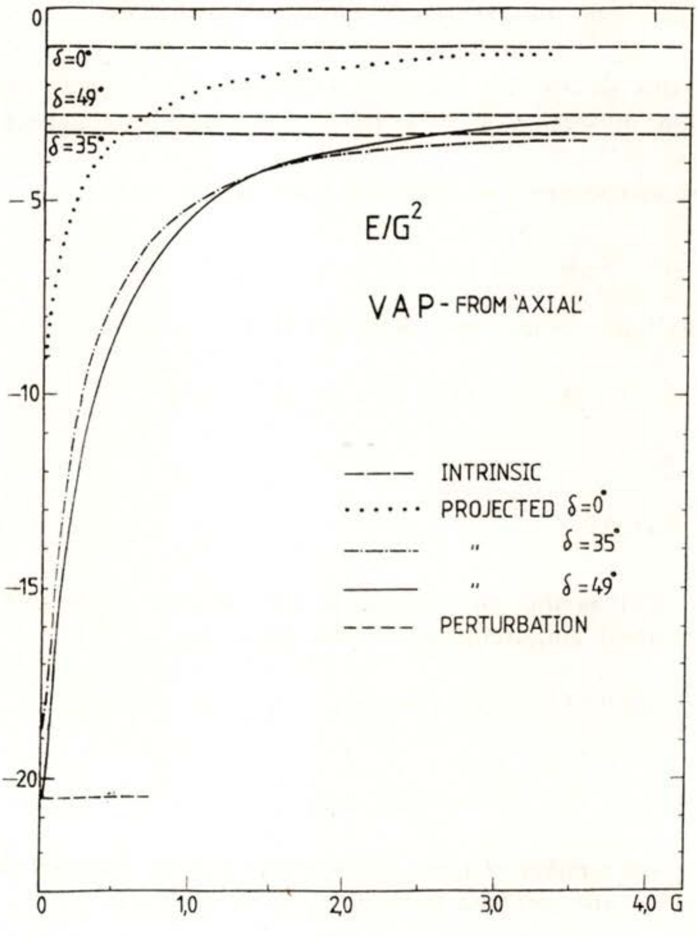

Fig. 5 — The quantity E/G^2 for the nucleon is plotted versus coupling constant for different values of the mixing angle. The physical states have been obtained from axially symmetric coherent states of pions. Here a scale different from the one of fig. 3 is used.

This agrees with (3.5), given by the perturbational calculation. Therefore one is led to the conclusion that the VAP from axially symmetric coherent states of pions should be a good approximation in the regimes of weak coupling strength. In regimes of very large G, the axially symmetric state considered in this section seems to be not so good: the energy (4.6) obtained in this limit is only one third of (3.7).

5 – PROJECTION FROM HEDGEHOG

In this section we consider a projection similar to the one presented in section 4, now from the hedgehog coherent state of pions.

Let us consider the following bare baryon state:

$$|B>_h = (1/\sqrt{2})\ (|N>_h + |\Delta>_h) \tag{5.1}$$

with the bare nucleon and delta states given by

$$|N>_h = (1/\sqrt{2})\ (|N^{+}_{-1/2}> - |N^{0}_{1/2}>) \tag{5.2a}$$

and

$$|\Delta>_h = (1/2)\ (|\Delta^{++}_{-3/2}> - |\Delta^{+}_{-1/2}> + |\Delta^{0}_{1/2}> - |\Delta^{-}_{3/2}>) \tag{5.2b}$$

Taking (5.1) as the bare baryon in the coherent state (3.8), the self-consistent amplitudes (3.14) are given by

$$\xi_{tm} = (\xi/\sqrt{3})\ (\delta_{t1}\ \delta_{m-1} + \delta_{t-1}\ \delta_{m1} - \delta_{t0}\ \delta_{m0})\ , \tag{5.3}$$

where

$$\xi = (9\sqrt{3}/5)\ G\ . \tag{5.4}$$

The average number of pions in the cloud and the intrinsic interaction energy are obtained from (3.11-12) and (5.3-4):

$$S = E^{(intr.)}_{kin} = \xi^2 = (243/25)\ G^2\ ; \tag{5.5a}$$

$$E^{(intr.)}_{coup} = -(18\sqrt{3}/5)\ G\,\xi = -(486/25)\ G^2\ . \tag{5.5b}$$

The coherent state (3.8) now reads as

$$|\,\mathrm{Hh} > = \mathcal{N}(\xi)\ \exp\,[(\xi/\sqrt{3})\ (\overset{+}{a}_{1-1} + \overset{+}{a}_{-11} - \overset{+}{a}_{00})]\ |\,\mathrm{B} >_{\mathrm{h}} \qquad (5.6)$$

This state is known under the name 'hedgehog' and its energy, the sum of (5.5a) and (5.5b), is given by (3.7).

One should notice that the coherent state (5.6) contains bare nucleon and bare delta components exactly in the same proportion. It is interesting to study a generalized hedgehog coherent state which, like (4.7), allows different weights for the bare nucleon and the bare delta:

$$|\,\mathrm{Hh}(\delta) > = \mathcal{N}(\xi)\ \exp\,[(\xi/\sqrt{3})\ (\overset{+}{a}_{1-1} + \overset{+}{a}_{-11} - \overset{+}{a}_{00})]\,|\,\mathrm{B}(\delta) >_{\mathrm{h}}\ , \qquad (5.7)$$

where

$$|\,\mathrm{B}(\delta) >_{\mathrm{h}} = \cos\delta\ |\,\mathrm{N} >_{\mathrm{h}} + \sin\delta\ |\,\Delta >_{\mathrm{h}}\ . \qquad (5.8)$$

The self-consistent pion amplitudes (3.14) are identical to (5.3) but now

$$\xi = \mathrm{G}\ \sqrt{3}\ (1 + (8/5)\ \sin\delta\ \cos\delta)\ . \qquad (5.9)$$

The total intrinsic energy is

$$\mathrm{E}^{(\mathrm{intr.})} = -3\,\mathrm{G}^2\ (1 + (8/5)\ \sin\delta\ \cos\delta)^2\ . \qquad (5.10)$$

Minimization with regard to δ yields $\delta = 45^{\circ}$, i. e. the result (3.7) is recovered, according to ref. [8]. Obviously, if (5.7) is projected onto states with good isospin and angular momentum quantum numbers and, afterwards, the projected energies are varied with respect to the mixing angle, then δ will take values in general different from 45°.

The generalized hedgehog coherent state has grand spin zero, i. e. it only contains components with $\mathrm{J} = \mathrm{T}$ and $\mathrm{M} = -\mathrm{M}_{\mathrm{T}}$. Therefore it is enough to perform the projection restricted to one space [8, 11]. The spin and isospin eigenstates are

$$|\,\mathrm{J}\,;\ \mathrm{T} = \mathrm{J}\,;\ \mathrm{M}\,; -\mathrm{M} > = \mathrm{P}_{\mathrm{JM}}\,|\,\mathrm{Hh}(\delta) >\ , \qquad (5.11)$$

with P_{JM} the projector defined in Appendix B. There, the norms of the projected states are evaluated and the general expression for the kinetic energy of the projected states is derived. The evaluation of the expectation values for the coupling Hamiltonian is straightforward and the total projected energies read as

$$E_{1/2} = (\xi^2/3)\ [1 + f_{3/2}(\delta, \xi^2)/f_{1/2}(\delta, \xi^2)]$$
$$- 2G\sqrt{3}\,\xi\ [f_{1/2}(\delta, \xi^2) + (4/5)\ \sin\delta\ \cos\delta\ (f^N_{1/2} + f^\Delta_{1/2})]/f_{1/2}(\delta, \xi^2) \tag{5.12}$$

$$E_{3/2} = (\xi^2/3)\ [1 + (f_{1/2}(\delta, \xi^2) + f_{5/2}(\delta, \xi^2))/f_{3/2}(\delta, \xi^2)]$$
$$- 2G\sqrt{3}\,\xi\ [f_{3/2}(\delta, \xi^2) + (4/5)\ \sin\delta\ \cos\delta\ (f^N_{3/2} + f^\Delta_{3/2})]/f_{3/2}(\delta, \xi^2) \tag{5.13}$$

where f_J, f^N_J and f^Δ_J are given by (B. 6,14).

For $\delta = 45°$ these are the projected energies obtained from the normal hedgehog (5.6).

5.1 — *Variation-before-projection*

In a VBP calculation from the normal hedgehog, one gets the projected numbers of pions and energies, putting $\delta = 45°$ in (B. 20) and (5.12-13) respectively, and substituting ξ by its self-consistent value (5.4). For the nucleon, the projected number of pions is shown in Fig. 6, in dependence of G. The dashed curve in Fig. 7 refers to the quantity $E_{1/2} / G^2$ in a VBP from normal hedgehog. Figs. 6 and 7 also display the number of pions and $E_{1/2} / G^2$ obtained in the MFA and VAP.

In the limit $G \rightarrow 0$ one gets $E_{1/2} / G^2 = -14.58$, a value which is very far from the one given by (3.5) and well recovered in the VAP-from 'axial'.

5.2 — *Variation-after-projection*

In a VAP from normal hedgehog, the amplitude ξ is determined by means of a Ritz's variation of the projected energies (5.12-13) with $\delta = 45^{o}$. For the numerical evaluation of the projected quantities we have used the procedure already explained in the previous section when the VAP-from 'axial' was considered.

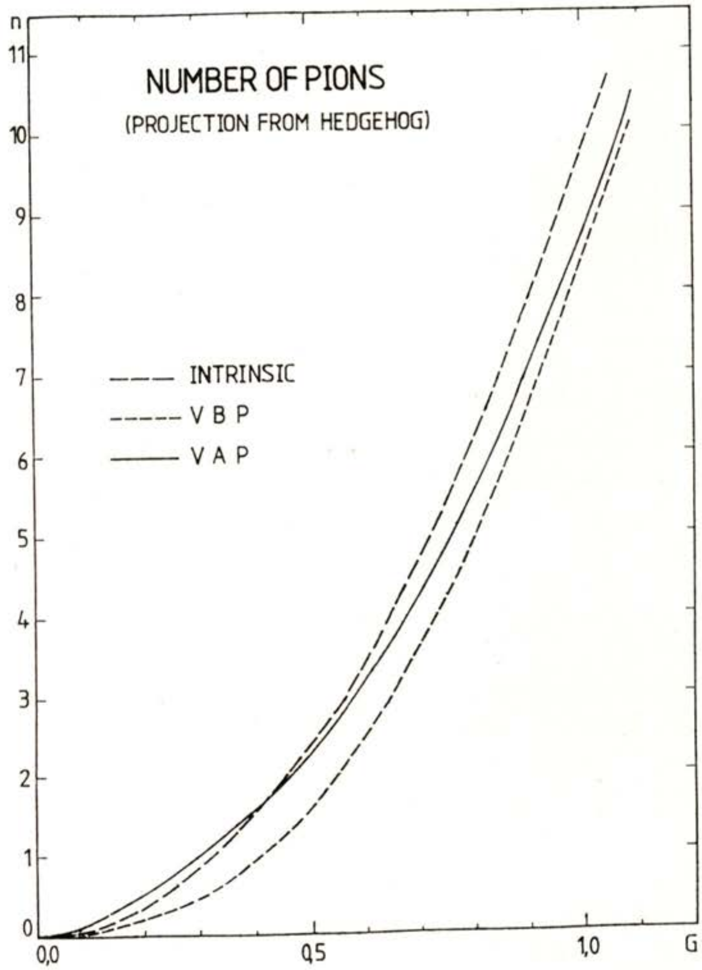

Fig. 6 — The number of pions in the pion cloud of the nucleon. The curves refer to the intrinsic hedgehog state and to the projected nucleon obtained from the hedgehog in VAP and VBP calculations.

The projected nucleon energy and number of pions are shown in Figs. 6 and 7 (solid lines). In the limit $G \rightarrow 0$ one gets $E_{1/2} = -19.44\ G^2$ for the nucleon and $E_{3/2} = -9.72\ G^2$ for the delta, i. e. the perturbative results given by (3.5-6) are almost reached. However, a full VAP calculation using the delta dependent hedgehog coherent state (5.7) improves the results obtained from

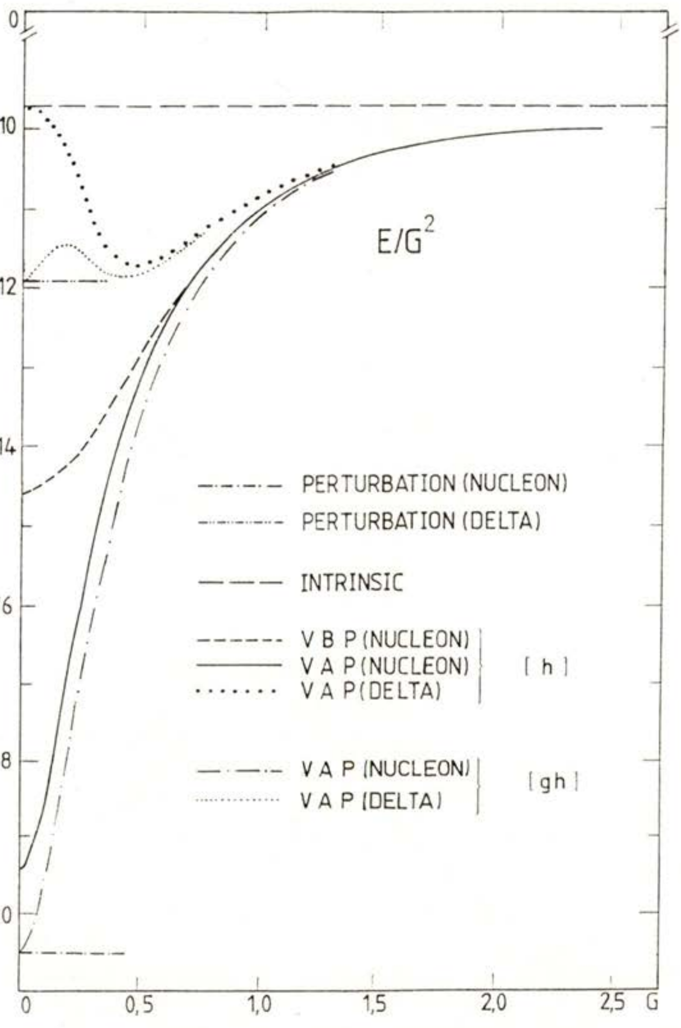

Fig. 7 — For the nucleon and the delta, the quantities E/G^2 obtained in a projection from the hedgehog state (5.6), [h], and from the general hedgehog (5.7), [gh], are plotted versus coupling constant.

normal hedgehog, as it is shown in Fig. 7, in the whole range of G. In particular, the energies of the nucleon and the delta in the limit $G \rightarrow 0$, turn out to be exactly the perturbative values (3.5-6), for mixing angles $\delta = 58^{\circ}$ (nucleon) and $\delta = 68.2^{\circ}$ (delta).

In the limit $G \rightarrow \infty$, when a large number of pions is present in the cloud, the best value for δ is 45° in both cases. In this region the projection does not alter the MFA results, i. e. the nucleon and its isobars form a rotational band with a very large moment of inertia.

6 – CONCLUSIONS

In this paper we have studied different approximation schemes to the Hamiltonian for a system of non-self-interacting pions coupled to a static source with bare nucleon and bare delta.

In a realistic calculation, the approximation more suitable depends strongly on the size of source and the strength of the pion field. Here, the number of pions is a decisive parameter. We emphasize the usefulness of the simple model: for a given number of pions obtained in a realistic calculation one can read the bare coupling constant G from Figs. 1, 2, 4 or 6 and then, from Figs. 3, 5 and 7 one may compare the behaviour of the different approximations studied in this work, for that range of G.

We shall briefly discuss and compare the approximations considered in this work. The VAP is the right procedure to introduce quantum fluctuations in the mean-field or coherent state description of the pion cloud. The importance of this method is clearly demonstrated comparing the curves displayed in Figs. 3 and 5 (axial symmetric coherent state), on the one hand and Fig. 7 (hedgehog coherent state), on the other hand, for small values of G. From Figs. 3 and 5 one also concludes that the inclusion of bare delta in the intrinsic states, allowing for some nucleon-delta transitions, is very important.

The perturbative energy for the nucleon is exactly reproduced in the VAP-from 'axial' calculation, but a similar agreement with the energy (3.7) is not observed for $G \rightarrow \infty$. The hedgehog apparently is a much better ansatz. We reemphasize the following remarkable fact: using the generalized hedgehog one obtains the exact results (3.5-7), which refer to both the limits of very weak and very strong couplings, either for nucleon or delta.

Finally we refer a technical aspect: all projected quantities (energy, number of pions, etc.) may be expressed as functions of the norms, or their derivatives, of the projected states. For the axial state, these norms are given in terms of power series expansions which converge very fast. For the hedgehog they are expressed as functions of the modified Bessel functions $I_\nu(z)$, which are easily evaluated numerically by rapidly converging series in z or 1/z.

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APPENDIX A – THE NORM OF THE PROJECTED COHERENT STATE (AXIAL SYMMETRY)

From the intrinsic kets

$$|\psi_N> = \exp(\sqrt{S}\ \overset{+}{a}_{00})\ |\overset{+}{N}_{1/2}> \tag{A.1}$$

and

$$|\psi_\Delta> = \exp(\sqrt{S}\ \overset{+}{a}_{00})\ |\overset{+}{\Delta}_{1/2}>\ , \tag{A.2}$$

states with the quantum numbers of a nucleon are obtained according to

$$P_{1/2\ 1/2}|\psi_N> = \iint d(\cos\beta)\, d(\cos\tilde{\beta})\, d^{1/2}_{1/2\ 1/2}(\beta)\, d^{1/2}_{1/2\ 1/2}(\tilde{\beta})\, R(\beta)\, R(\tilde{\beta})\, |\psi_N> \tag{A.3}$$

and

$$P_{1/2\ 1/2}|\psi_\Delta> = \iint d(\cos\beta)\, d(\cos\tilde{\beta})\, d^{1/2}_{1/2\ 1/2}(\beta)\, d^{1/2}_{1/2\ 1/2}(\tilde{\beta})\, R(\beta)\, R(\tilde{\beta})\, |\psi_\Delta> \tag{A.4}$$

$P_{1/2\,1/2}$ is a projector with the properties: $P^2_{1/2\,1/2} = P^+_{1/2\,1/2} = P_{1/2\,1/2}$ The operator $R(\beta)$ rotates the unprojected state by an angle β around the y-axis and $R(\tilde{\beta})$ performs a similar rotation in isospin.

The overlap integrals are defined and evaluated as in ref. [5, 6, 9, 10]:

$$F_N(S) = \langle \psi_N | P_{1/2\,1/2} | \psi_N \rangle$$

$$= \iint d(\cos\beta)\, d(\cos\tilde{\beta})\, d^{1/2}_{1/2\,1/2}(\beta)\, d^{1/2}_{1/2\,1/2}(\tilde{\beta})\, d^{1/2}_{1/2\,1/2}(\beta)$$

$$d^{1/2}_{1/2\,1/2}(\tilde{\beta})\; e^{S\cos\beta\cos\tilde{\beta}}; \qquad (A.5)$$

$$F_\Delta(S) = \langle \psi_\Delta | P_{1/2\,1/2} | \psi_\Delta \rangle$$

$$= \iint d(\cos\beta)\, d(\cos\tilde{\beta})\, d^{1/2}_{1/2\,1/2}(\beta)\, d^{1/2}_{1/2\,1/2}(\tilde{\beta})\, d^{3/2}_{1/2\,1/2}(\beta)$$

$$d^{3/2}_{1/2\,1/2}(\tilde{\beta})\; e^{S\cos\beta\cos\tilde{\beta}}. \qquad (A.6)$$

The most practical way is to evaluate these integrals by power series expansion which converges rapidly for $S \lesssim 50$. By calling $\cos\beta = x$ and $\cos\tilde{\beta} = y$, we get

$$F_N(S) = \int_{-1}^{+1}\int_{-1}^{+1} dxdy\,(1+x)/2 \cdot (1+y)/2 \cdot [1 + Sxy +$$
$$(Sxy)^2/2! + \ldots] = f(S) + f'(S) \qquad (A.7)$$

where

$$f(S) = \sum_{n=0}^{\infty} \frac{S^{2n}}{(2n)!\,(2n+1)^2} \qquad (A.8)$$

and $f'(S)$ denotes derivative with respect to S. Similarly we get

$$F_\Delta(S) = \int_{-1}^{+1}\int_{-1}^{+1} dxdy\,(1+x)/2 \cdot (1+y)/2 \cdot (3x-1)/2 \cdot$$

$$\cdot (3y-1)/2 \cdot [1 + Sxy + (Sxy)^2/2! + \ldots]$$

$$= (1/4)\, f(S) + f'(S) + (9/4)\, f''(S) + g(S), \qquad (A.9)$$

where

$$g(S) = -3/2 \sum_{n=0}^{\infty} \frac{S^{2n}}{(2n)!(2n+1)(2n+3)} \ . \qquad (A.10)$$

For the state

$$|\psi> = \cos\delta\,|\psi_N> + \sin\delta\,|\psi_\Delta> , \qquad (A.11)$$

the overlap integral is

$$F = <\psi|P_{1/2\,1/2}|\psi> = \cos^2\delta\ F_N + \sin^2\delta\ F_\Delta \qquad (A.12)$$

In order to calculate the number of pions in the projected state, we take into account the relation

$$\frac{\partial|\psi>}{\partial S} = \frac{1}{2\sqrt{S}}\ a^+_{00}|\psi> = \frac{1}{2S}\ a^+_{00}\ a_{00}|\psi> , \qquad (A.13)$$

to evaluate the derivative of the overlap integral F. This gives

$$\begin{aligned} dF/dS &= d<\psi|P_{1/2\,1/2}|\psi>/dS \\ &= (1/S) <\psi|P_{1/2\,1/2}\,a^+_{00}\,a_{00}|\psi> \ . \end{aligned} \qquad (A.14)$$

The projected number of pions is easily obtained:

$$\begin{aligned} n &= F^{-1} <\psi|P_{1/2\,1/2} \sum_{tm} a^+_{tm}\ a_{tm}\ P_{1/2\,1/2}|\psi> \\ &= F^{-1} <\psi|P_{1/2\,1/2} \sum_{tm} a^+_{tm}\ a_{tm}|\psi> \\ &= F^{-1} <\psi|P_{1/2\,1/2}\ a^+_{00}\ a_{00}|\psi> = (S/F)\ dF/dS , \end{aligned} \qquad (A.15)$$

since only $a_{00}|\psi>$ in non-zero.

APPENDIX B – THE NORM OF THE PROJECTED COHERENT STATE (HEDGEHOG)

The generalized hedgehog coherent state is defined by

$$|Hh(\delta)> = \cos\delta\,|\tilde{N}>_h + \sin\delta\,|\tilde{\Delta}>_h \qquad (B.1)$$

where

$$| \tilde{N} >_h = \exp [\sqrt{S}/3 \ (a^+_{1-1} + a^+_{-11} - a^+_{00})] \, | N >_h \qquad (B.2)$$

and

$$| \tilde{\Delta} >_h = \exp [\sqrt{S}/3 \ (a^+_{1-1} + a^+_{-11} - a^+_{00})] \, | \Delta >_h ; \qquad (B.3)$$

$| N >_h$ and $| \Delta >_h$ are defined by (5.2). The state (B.1) has grand spin zero:

$$(\mathbf{J} + \mathbf{T}) \, | Hh (\delta) > = 0 , \qquad (B.4)$$

i. e. it only contains components with $J = T$ and $M_T = - M$. Thus, separate projections for spin and isospin are not required. The eigenstates of spin and isospin are obtained from (B.1) according to [11]

$$| J , T = J ; M , - M > = P_{JM} \, | Hh (\delta) >$$
$$= (2J + 1) / 8\pi^2 \int d^3 \Omega \, \mathcal{D}^J_{MM} (\Omega) \, R (\Omega) \, | Hh (\delta) > . \qquad (B.5)$$

Here Ω represents the three Euler angles α, β and γ and $R (\Omega)$ stands for the rotation operator. The operator P_{JM} is a projector: $P^2_{JM} = P^+_{JM} = P_{JM}$.

We shall always consider $M = J$; the norm of this state is

$$f_J (\delta , S) = < Hh (\delta) \, | P_{JJ} | \, Hh (\delta) >$$
$$= \cos^2 \delta \ f^N_J (S) + \sin^2 \delta \ f^\Delta_J (S) , \qquad (B.6)$$

where

$$f^N_J (S) = (2J + 1) / 8\pi^2 \int d^3 \Omega \, \mathcal{D}^J_{JJ} (\Omega) \, \mathcal{H}_N (\Omega , S) \qquad (B.7a)$$

$$f^\Delta_J (S) = (2J + 1) / 8\pi^2 \int d^3 \Omega \, \mathcal{D}^J_{JJ} (\Omega) \, \mathcal{H}_\Delta (\Omega , S) . \qquad (B.7b)$$

In these expressions, the kernels $\mathcal{H}_N$ and $\mathcal{H}_\Delta$ are given by

$$\mathcal{H}_N (\Omega , S) = {}_h < \tilde{N} \, | R (\Omega) | \, \tilde{N} >_h$$
$$= \mathcal{H}_\pi (\Omega , S) \ \cos \beta/2 \ \cos (\alpha + \gamma) / 2 \qquad (B.8a)$$

and

$$\mathcal{N}_{\Delta}(\Omega, S) = {}_{h}\langle \tilde{\Delta} | R(\Omega) | \tilde{\Delta} \rangle_{h}$$
$$= \mathcal{N}_N(\Omega, S)\ (2\cos^2\beta/2\ \cos^2(\alpha+\gamma)/2 - 1), \qquad \text{(B.8b)}$$

where

$$\mathcal{N}_{\pi}(\Omega, S) = \exp\{ S/3\,[(1+\cos\beta)\cos(\alpha+\gamma) + \cos\beta]\} \qquad \text{(B.9)}$$

refers to the overlap between rotated pion clouds. To calculate the norms (B.7) we use the expression

$$\mathcal{D}^{J}_{JJ}(\Omega) = e^{-iJ(\alpha+\ \ )}(\cos\beta/2)^{2J} \qquad \text{(B.10)}$$

and define the following new variables: $z = 2S/3$, $x = \cos^2\beta/2$, $k = J\text{-}1/2$ and $\varphi = \alpha+\gamma$ ($d\alpha d\gamma = d\alpha d\varphi$, and the integral over α is trivial, giving 2π). The expressions for f^N_J and f^Δ_J reduce to

$$f^N_J = (k+1)\ e^{-z/2}\int_0^1 dx\ x^{k+1}\ e^{zx}$$
$$\left\{\frac{1}{2\pi}\int_0^{2\pi} d\varphi\ [\cos k\varphi + \cos(k+1)\varphi]\ e^{zx\cos\varphi}\right\} \qquad \text{(B.11a)}$$

$$f^\Delta_J = -f^N_J + \frac{k+1}{2}\ e^{-z/2}\int_0^1 dx\ x^{k+2}\ e^{zx}\left\{\frac{1}{2\pi}\int_0^{2\pi} d\varphi\ e^{zx\cos\varphi}\right.$$
$$\left.\cdot[\cos(k-1)\varphi + 3\cos k\varphi + 3\cos(k+1)\varphi + \cos(k+2)\varphi]\right\}$$
$$\text{(B.11b)}$$

and we recognize the integrals over φ as the modified Bessel functions of integer order $I_\nu(t)$ (see (9.6.19) of ref. [12]):

$$f^N_J = (k+1)\,e^{-z/2}\ z^{-(k+2)}\int_0^z dt\ e^t\ t^{k+1}\ [I_k(t) + I_{k+1}(t)]$$
$$\text{(B.12a)}$$

$$f^\Delta_J = -f^N_J + \frac{k+1}{2}\ e^{-z/2}\ z^{-(k+3)}\int_0^z dt\ e^t\ t^{k+2}$$
$$[I_{k-1}(t) + 3I_k(t) + 3I_{k+1}(t) + I_{k+2}(t)], \qquad \text{(B.12b)}$$

where $t = xz$. Defining the function $g_{\mu,\nu}(z)$ (see (11.3.1) of ref. [12], with $p = -1$) as

$$g_{\mu,\nu}(z) = \int_0^z dt\, e^t\, t^{\mu}\, I_\nu(t) \qquad \text{(B.13)}$$

the above expressions for the overlaps read as

$$f_J^N = (k+1)\, e^{-z/2} z^{-(k+2)} [g_{k+1,k}(z) + g_{k+1,k+1}(z)] \qquad \text{(B.14a)}$$

$$f_J^\Delta = -f_J^N + (k+1)\, e^{-z/2} z^{-(k+3)} [g_{k+2,k+1}(z) + g_{k+2,k+2}(z) + (k/2)\, g_{k+1,k}(z) + 3(k+1)/2\; g_{k+1,k+1}(z)]\,. \qquad \text{(B.14b)}$$

In order to write f_J^Δ in this form, we have used the recursion relation for the modified Bessel functions ((9.6.26) of ref. [12]) and the definition (B.13) of the g-functions.

The expressions in brackets in (B.14) can be written as functions of $I_\nu(z)$. Using the formulas (11.3.3,6,12) and (9.6.26) of ref. [12], one gets

$$f_J^N(S) = (1/z)\, \exp(z/2) \cdot (k+1)\; I_{k+1}(z) \qquad \text{(B.15a)}$$

$$f_J^\Delta(S) = 1/(2z)\, \exp(z/2)\, [k\, I_k(z) + (k+2)\, I_{k+2}(z)] \qquad \text{(B.15b)}$$

where $k = J - 1/2$ and $z = 2S/3$.

The modified Bessel functions $I_\nu(z)$ are evaluated by rapidly converging power series in z, or, for $z > 3.75$, by the asimptotic series in z^{-1} (see ref. [12]).

In order to evaluate the kinetic energy one should note that, for the state (B.1) the property (3.9) reads as

$$< Hh(\delta)\,|\, a_{tm}^{+} = \sqrt{S/3}\; \delta_{t\text{-}m}\,(-1)^{t+1} < Hh(\delta)\,|\,. \qquad \text{(B.16)}$$

The projected kinetic energy is

$$E_{kin}^J = f_J^{-1}(\delta, S) < Hh(\delta)\,|\, P_{JJ} \sum_{tm} a_{tm}^{+} a_{tm} P_{JJ}\,|\, Hh(\delta) >\,, \qquad \text{(B.17)}$$

where $f_J(\delta, S)$ is defined by (B.6). The first projector can be commuted to the right of the kinetic energy operator and dropped. Next we use the property (B.16) and take into account the commutation relation of P_{JJ} with the operator a_{t-t} (see Appendix B of ref. [6]):

$$a_{t-t} P_{JJ} = \sum_{J'} P_{J' J-t} \, a_{t-t} \, [\mathcal{C}^{JJ}_{J' J-t 1 t}]^2 + \text{other terms}, \qquad (B.18)$$

where the sum is over all values of J′ such that

$$| J' - 1 | \leqslant J \leqslant | J' + 1 | . \qquad (B.18a)$$

In (B.18) the 'other terms' contain the operator a_{tm} with $t \neq -m$ which gives zero acting on the hedgehog ket according to (B.16) and therefore can be ignored. Noticing that $< Hh(\delta) \, | P_{J' J-t} | \, Hh(\delta) >$ is independent of J - t and using the property of the Clebsh-Gordan coefficients

$$\sum_t [\mathcal{C}^{JJ}_{J' J-t 1 t}]^2 = 1 , \qquad (B.19)$$

one gets

$$E^J_{kin} = (S/3) \, f_J^{-1}(\delta, S) \sum_{J'} f_{J'}(\delta, S) . \qquad (B.20)$$

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